**18A The Area Under a Curve**

Consider the function $y = 5$. Find the area under the curve for $1 \le x \le 4$.


- rectangle

$A = l \times w$

$= (5)(3)$

$= 15$

Now evaluate $\int_1^4 5\,dx$

$\int_1^4 5dx = 5x\Big]_1^4$

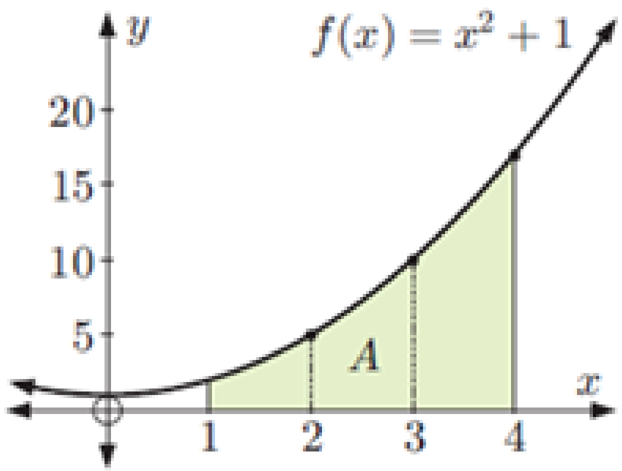
$= (5(4)) - (5(1))$

$= 20 - 5$

$= 15$

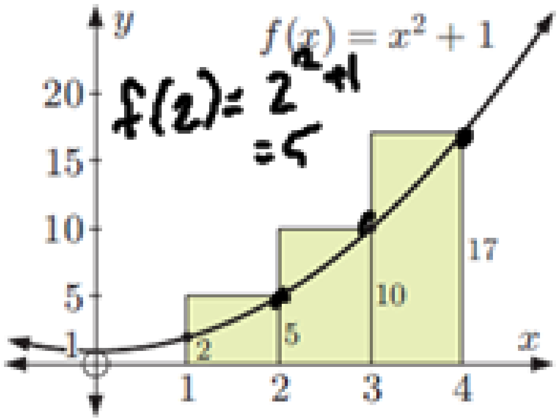
Next, consider the function $f(x) = x^2 + 1$

Want to estimate the area under the curve for $1 \leq x \leq 4$.
Divide the interval into three strips of equal width:




Extend these strips to make rectangles.

## Upper Rectangles:




What is the area?

$A_T = A_1 + A_2 + A_3$

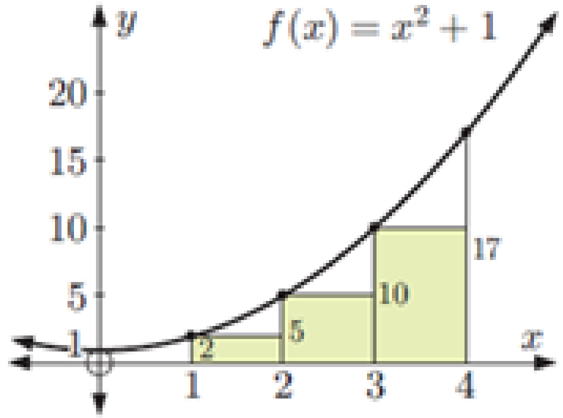
$= (5)(1) + (10)(1) + (17)(1)$

$= 5 + 10 + 17$

$= 32$

overestimate

## Lower Rectangles:




What is the area?

$A_T = A_1 + A_2 + A_3$

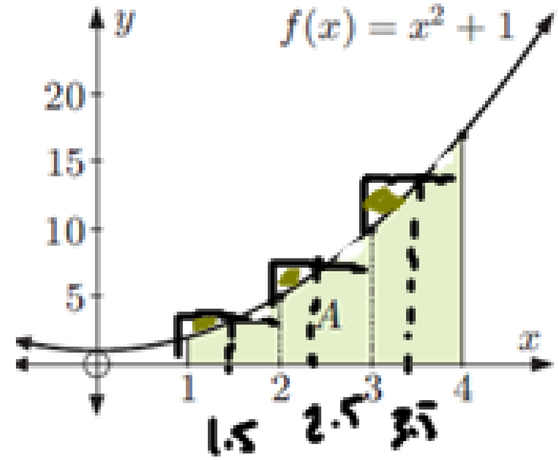
$= (2)(1) + (5)(1) + (10)(1)$

$= 2 + 5 + 10$

$= 17$

under estimate

## Middle Rectangles




What is the area?

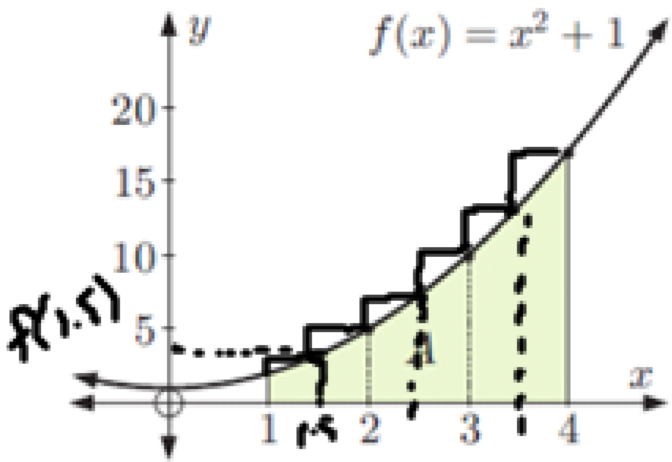
$A_T = A_1 + A_2 + A_3$

$= (3.25)(1) + (7.25)(1)$

$+ (13.25)(1)$

$= 23.75$

Now change the width of each rectangle to $\frac{1}{2}$, so there are 6 subintervals. Calculate the area of the upper and lower rectangles.




UPPER

$A_T = A_1 + A_2 + A_3 + A_4 + A_5 + A_6$

$= \frac{1}{2}((1.5)^2+1) + \frac{1}{2}(5) + \frac{1}{2}(7.25) + \frac{1}{2}(10)$

$+ \frac{1}{2}(13.25) + \frac{1}{2}(17)$

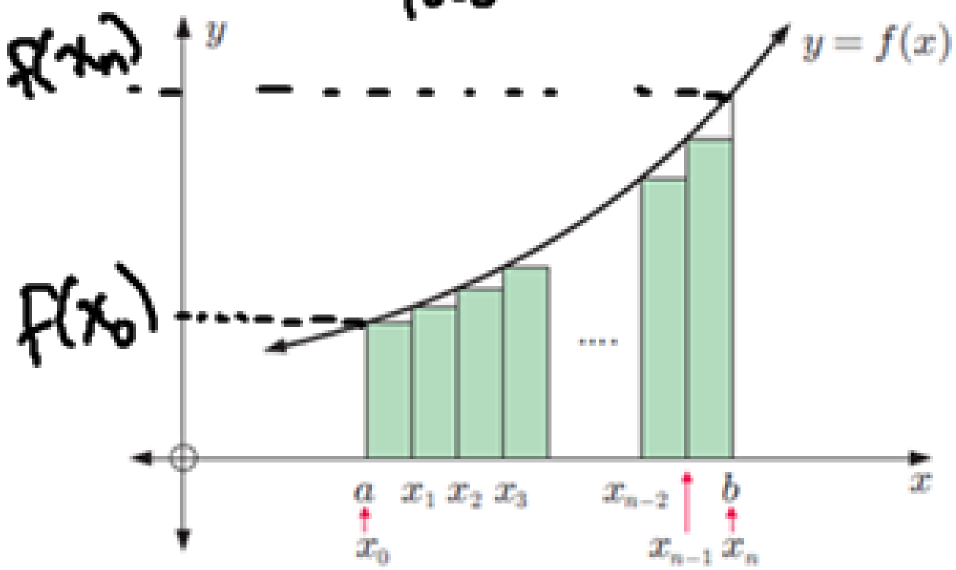
$= 27.875$

As more subintervals are created, the values of the lower and upper rectangle areas become closer and will converge to the value of $A$, the area under the curve.

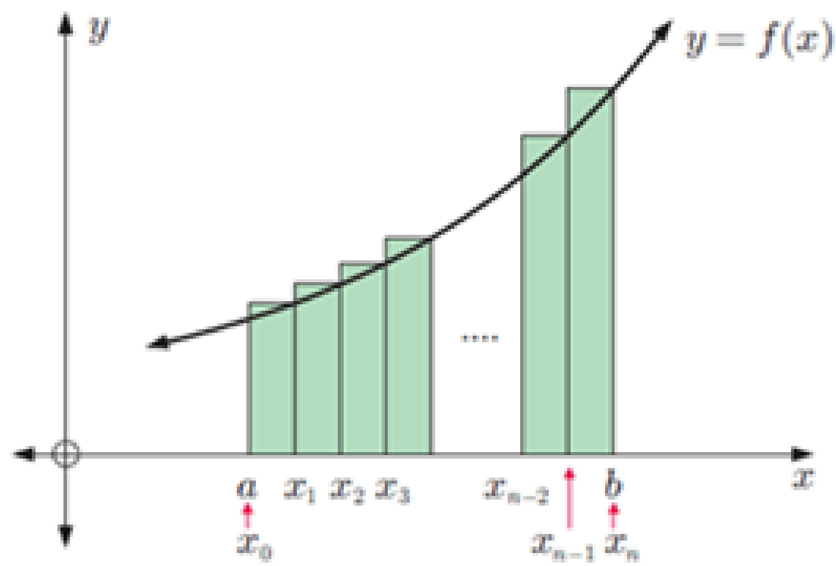
Consider the lower and upper rectangle sums for a function which is positive and increasing on the interval $a \leq x \leq b$. Divide the interval into $n$ subintervals, each of width

$$w = \frac{b-a}{n}.$$

lower




upper



The area of the lower and upper rectangles are:

Lower

$$A_T = A_1 + A_2 + A_3 + \cdots + A_n$$
$$= wf(x_0) + wf(x_1) + \cdots + wf(x_{n-1})$$
$$= w[f(x_0) + f(x_1) + \cdots + f(x_{n-1})]$$
$$= w\sum_{i=0}^{n-1} f(x_i)$$

upper

$$A_T = A_1 + A_2 + \cdots + A_n$$
$$= wf(x_1) + wf(x_2) + \cdots + wf(x_n)$$
$$= w(f(x_1) + f(x_2) + \cdots f(x_n))$$
$$= w\sum_{k=1}^{n} f(x_k)$$

Find:

$$A_U - A_L = w\sum_{k=1}^{n} f(x_k) - w\sum_{i=0}^{n-1} f(x_i)$$
$$= w\, f(x_n) - wf(x_0)$$
$$= w\,(f(x_n) - f(x_0))$$
$$= \frac{b-a}{n}(f(b) - f(a))$$

$$\lim_{n\to\infty}(A_u - A_L) = \lim_{n\to\infty} \frac{b-a}{n}\left[f(b)-f(a)\right]$$

$$= 0$$

as $n \to \infty$ $\quad A_u - A_L = 0 \quad A_u = A_L$

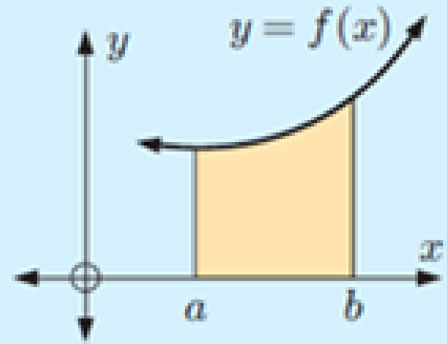
$$A = \lim_{n\to\infty} A_L$$

$$= \lim_{n\to\infty} \sum_{i=0}^{n-1} f(x_i)\,\omega \quad \leftarrow \text{width}$$

$$= \lim_{n\to\infty} \sum_{i=0}^{n-1} f(x_i)\Delta x = \int_a^b f(x)\,dx$$

# Fundamental Theorem of Calculus (18C)

If $f(x)$ is a continuous positive function on an interval $a \leqslant x \leqslant b$ then the area under the curve between $x = a$ and $x = b$ is $\int_a^b f(x)\,dx$.




For a continuous function $f(x)$ with antiderivative $F(x)$, $\int_a^b f(x)\,dx = F(b) - F(a)$.

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**3** Use the fundamental theorem of calculus to find the area between the $x$-axis and:

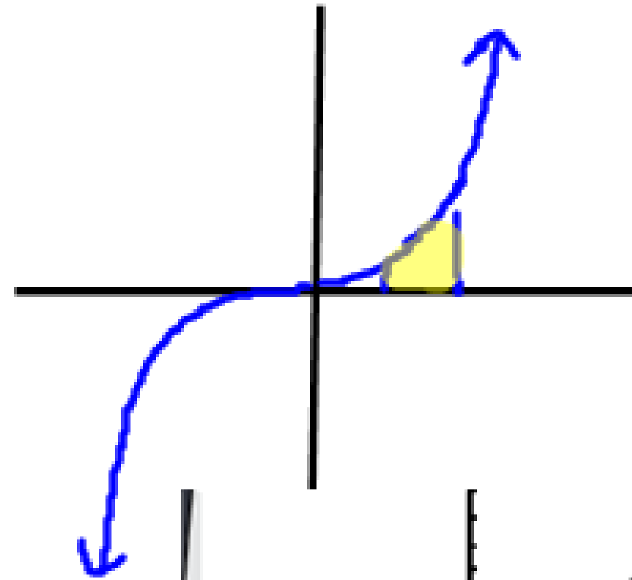
**a** $y = x^3$ from $x = 1$ to $x = 2$

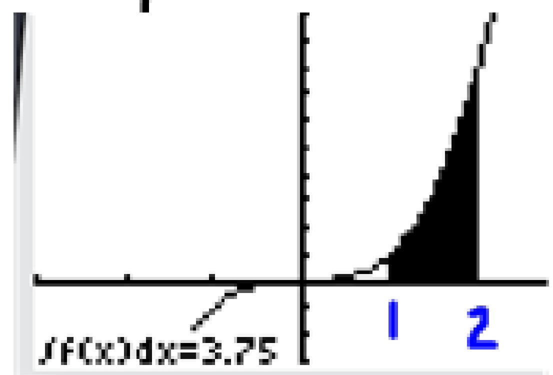
$$A = \int_1^2 x^3\,dx$$

$$= \frac{1}{4}x^4\Big]_1^2$$

$$= \frac{1}{4}(2)^4 - \frac{1}{4}(1)^4$$

$$= \frac{16}{4} - \frac{1}{4} = \frac{15}{4}$$







$\leftarrow$ $\boxed{\text{math}}$ $\boxed{9}$

$\rightarrow$ graph function $y = x^3$

$\rightarrow$ $\boxed{\text{2nd}}$ $\boxed{\text{trace}}$ $\boxed{7}$

**e** $y = \dfrac{1}{\sqrt{x}}$ from $x = 1$ to $x = 4$

$$A = \int_1^4 \frac{1}{\sqrt{x}}\,dx$$

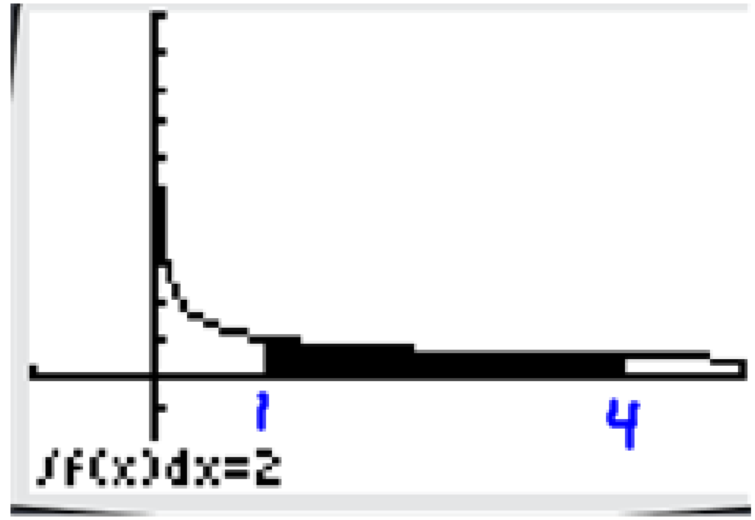
$$= \int_1^4 x^{-\frac{1}{2}}\,dx$$

$$= 2x^{\frac{1}{2}}\Big]_1^4$$

$$= 2\sqrt{4} - 2\sqrt{1}$$

$$= 4 - 2$$

$$= 2$$




HW pg 453 ch18C

\# 1, 3, 4, 5,