# Ch 15 H- Second and Higher Derivatives

First Derivative: $f'(x)$ or $\frac{dy}{dx}$

Second Derivative: The derivative of the derivative.

$$f''(x) \text{ or } \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$$

The $n^{th}$ Derivative: $f^{(n)}(x)$ or $= \frac{d^n y}{dx^n}$

Example: Find the second derivative of $y = 2x^3 - 5x$

$y' = 6x^2 - 5$

$y'' = 12x$

each time we take the derivative, the degree of the polynomial decreases by 1.

…Third derivative

$y''' = 12$

$y'''' = 0$

$y^{(5)} = 0$

Example: Given $y = \frac{x+1}{x^2}$ find $\frac{d^2y}{dx^2}$

Quotient Rule

$u = x+1 \qquad v = x^2$

$u' = 1 \qquad v' = 2x$

$$\frac{dy}{dx} = \frac{vu' - uv'}{v^2}$$

$$= \frac{(x^2)(1) - (x+1)(2x)}{(x^2)^2}$$

$$= \frac{x^2 - 2x^2 - 2x}{x^4}$$

$$\frac{dy}{dx} = \frac{-x^2 - 2x}{x^4}$$

$$\frac{dy}{dx} = \frac{-x-2}{x^3}$$

$\frac{d^2y}{dx^2}$ → Quotient Rule

$$\frac{dy}{dx} = \frac{-x-2}{x^3}$$

$\frac{d^2y}{dx^2}$ → Quotient Rule

$u = -x-2 \quad v = x^3$

$u' = -1 \quad v' = 3x^2$

$$\frac{d^2y}{dx^2} = \frac{vu' - uv'}{v^2}$$

$$= \frac{(x^3)(-1) - (-x-2)(3x^2)}{(x^3)^2}$$

$$= \frac{-x^3 + 3x^2(x+2)}{x^6}$$

Example: Given $y = (x^2 - 4)^3$ find $\frac{d^2y}{dx^2}$

Chain Rule

$y' = 3(x^2-4)^2(2x)$

$y' = 6x(x^2-4)^2$

Product Rule + Chain Rule

$u = 6x \qquad v = (x^2-4)^2$

$u' = 6 \qquad v' = 2(x^2-4)'(2x)$

$v' = 4x(x^2-4)$

$y'' = uv' + vu'$

$y'' = (6x)(4x(x^2-4)) + (x^2-4)^2(6)$

Example: Given $y = 5e^{2x}$ , show that $\frac{d^2y}{dx^2} = 4y$

$y' = 5(e^{2x}(2))$

$y' = 10e^{2x}$

$y'' = 10(e^{2x}(2))$

$y'' = 20e^{2x}$

$= 4(5e^{2x})$

$= 4y$

$y = e^x$

$y' = e^x$

$y = e^u \quad u = f(x)$

$y' = e^u \cdot u'$

Example: Given $y = \cos x$ , find the $22^{nd}$ derivative

$y' = -\sin x$

$y'' = -\cos x$

$y''' = -(-\sin x)$

$= \sin x$

$y^{(4)} = \cos x$

$y^{(5)} = -\sin x$

The $4^{th}$ derivative is the same as the functions, the derivative repeats every $4^{th}$ derivative

$\frac{22}{4} = 5\frac{2}{4}$

The $22^{nd}$ derivative of $y = \cos x$ is $\frac{d^{22}y}{dx^{22}} = \frac{d^2y}{dx^2} = -\cos x$

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**5** Find $x$ when $f''(x) = 0$ for:

**a** $f(x) = 2x^3 - 6x^2 + 5x + 1$

$f'(x) = 6x^2 - 12x + 5$

$f''(x) = 12x - 12$

$0 = 12x - 12$

$1 = x$

**8** Suppose $f(x) = 2\sin^3 x - 3\sin x$.

**a** Show that $f'(x) = -3\cos x \cos 2x$.

$f(x) = 2(\sin x)^3 - 3\sin x$

$f'(x) = 2\left[3(\sin x)^2(\cos x)\right] - 3[\cos x]$

$f'(x) = 6\cos x \sin^2 x - 3\cos x$

$= 3\cos x(2\sin^2 x - 1)$

$= 3\cos x(-\cos 2x)$

$= -3\cos x \cos 2x$

looks real close to...

$\cos 2\theta = 1 - 2\sin^2\theta$

mult each term by $-1$

$-\cos 2\theta = -1 + 2\sin^2\theta$

$-\cos(2\theta) = 2\sin^2\theta - 1$

## EXERCISE 15H

**1** Find $f''(x)$ given that:

**a** $f(x) = 3x^2 - 6x + 2$ **b** $f(x) = \frac{2}{\sqrt{x}} - 1$ **c** $f(x) = 2x^3 - 3x^2 - x + 5$

**d** $f(x) = \frac{2 - 3x}{x^2}$ **e** $f(x) = (1 - 2x)^3$ **f** $f(x) = \frac{x + 2}{2x - 1}$

**2** Find $\frac{d^2y}{dx^2}$ given that:

**a** $y = x - x^3$ **b** $y = x^2 - \frac{5}{x^2}$ **c** $y = 2 - \frac{3}{\sqrt{x}}$

**d** $y = \frac{4 - x}{x}$ **e** $y = (x^2 - 3x)^3$ **f** $y = x^2 - x + \frac{1}{1 - x}$

**3** Given $f(x) = x^3 - 2x + 5$, find:

**a** $f(2)$ **b** $f'(2)$ **c** $f''(2)$ **d** $f^{(3)}(2)$

**4** Suppose $y = Ae^{kx}$ where $A$ and $k$ are constants.

Show that: **a** $\frac{dy}{dx} = ky$ **b** $\frac{d^2y}{dx^2} = k^2y$ **c** $\frac{d^3y}{dx^3} = k^3y$

**5** Find $x$ when $f''(x) = 0$ for:

**a** $f(x) = 2x^3 - 6x^2 + 5x + 1$ **b** $f(x) = \frac{x}{x^2 + 2}$

**6** Consider the function $f(x) = 2x^3 - x$.

Complete the following table by indicating whether $f(x)$, $f'(x)$, and $f''(x)$ are positive (+), negative (−), or zero (0) at the given values of $x$.

| $x$ | $-1$ | 0 | 1 |
|---|---|---|---|
| $f(x)$ | − | | |
| $f'(x)$ | | | |
| $f''(x)$ | | | |

**7** Given $f(x) = \frac{2}{3}\sin 3x$, show that $f^{(3)}(\frac{2\pi}{9}) = 9$.

**8** Suppose $f(x) = 2\sin^3 x - 3\sin x$.

**a** Show that $f'(x) = -3\cos x \cos 2x$.

**b** Find $f''(x)$.

**9** Find $\dfrac{d^2y}{dx^2}$ given:

**a** $y = -\ln x$ **b** $y = x\ln x$ **c** $y = (\ln x)^2$

**10** Given $f(x) = x^2 - \dfrac{1}{x}$, find: **a** $f(1)$ **b** $f'(1)$ **c** $f''(1)$ **d** $f^{(3)}(1)$

**11** If $y = 2e^{3x} + 5e^{4x}$, show that $\dfrac{d^2y}{dx^2} - 7\dfrac{dy}{dx} + 12y = 0$.

**12** If $y = \sin(2x + 3)$, show that $\dfrac{d^2y}{dx^2} + 4y = 0$.

**13** If $y = \sin x$, show that $\dfrac{d^4y}{dx^4} = y$.

**14** If $y = 2\sin x + 3\cos x$, show that $y'' + y = 0$ where $y''$ represents $\dfrac{d^2y}{dx^2}$.