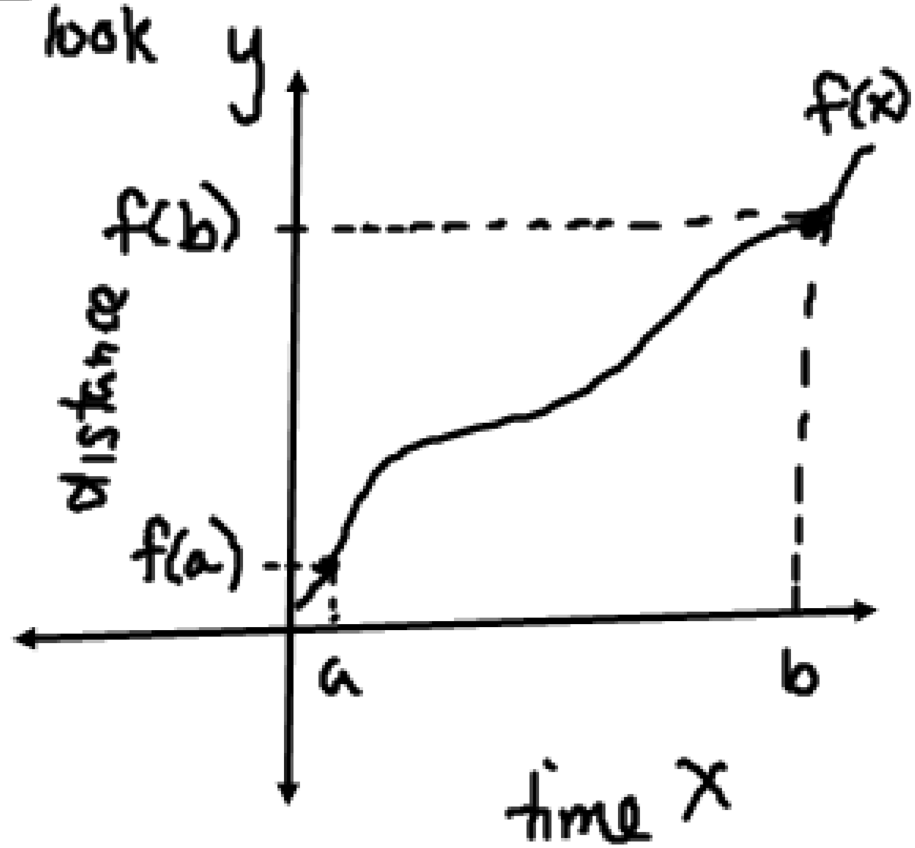
Ch 14C - Rates of Change

Average rate of change: The amount of change of a function divided by the length of the interval.

Ex: Average speed $= \dfrac{\text{how far you went}}{\text{how long it took}}$

$$= \frac{f(b)-f(a)}{b-a}$$

$$\text{Slope} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$



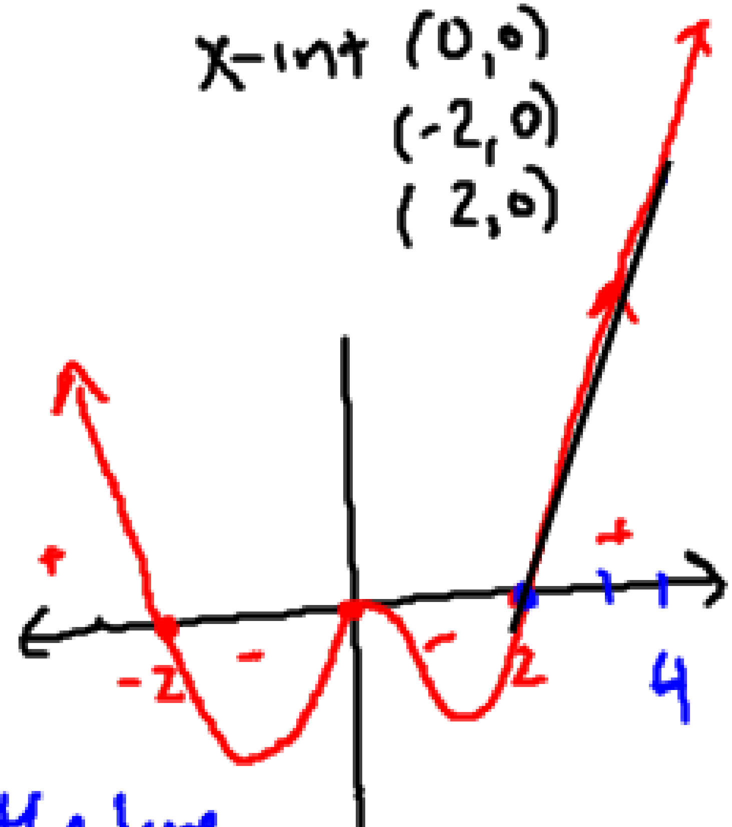
Example: Find the average rate of change of $f(x) = x^4 - 4x^2$ over the interval [2,4]

$$0 = x^4 - 4x^2$$
$$= x^2(x^2 - 4)$$
$$0 = x^2(x+2)(x-2)$$

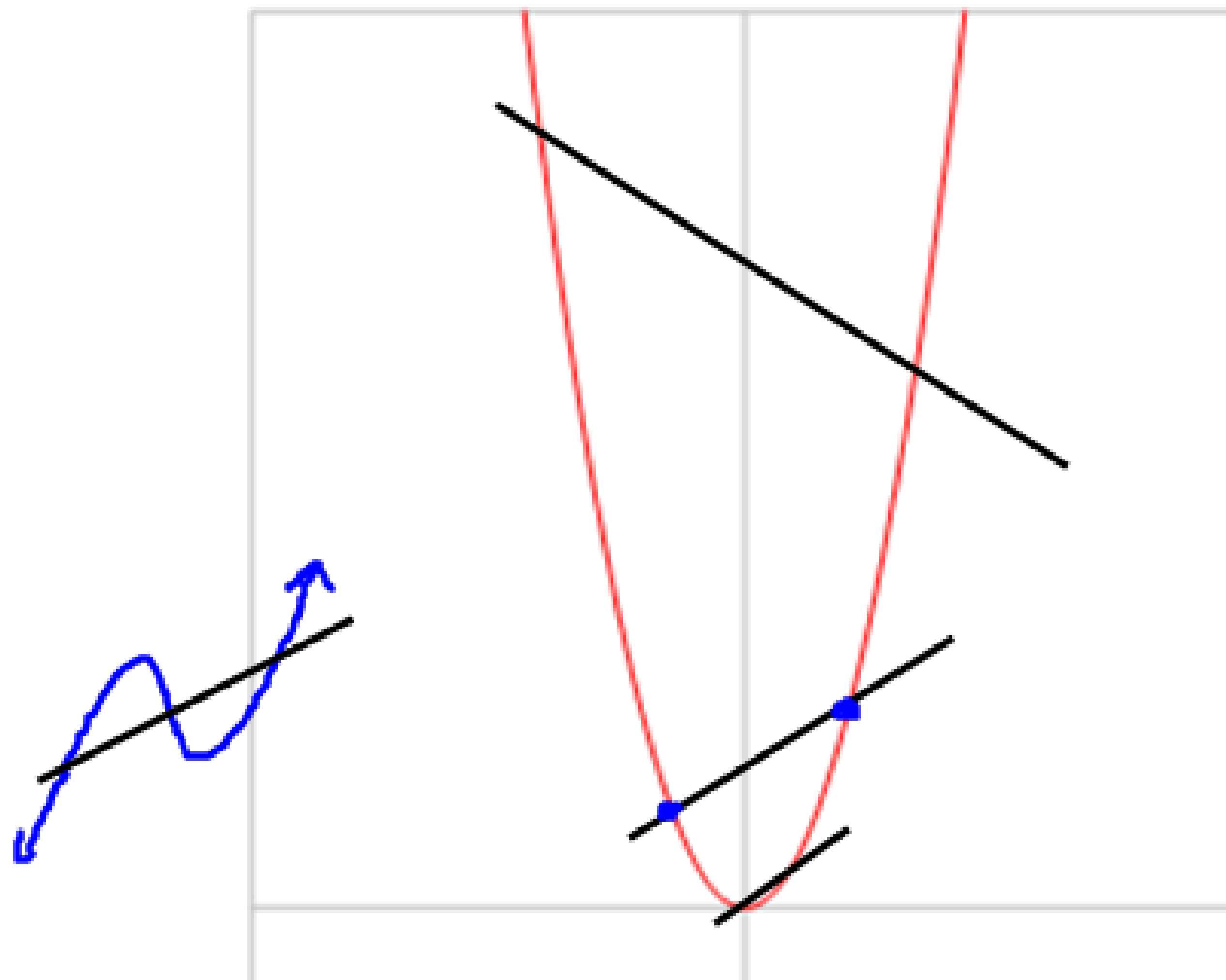
x-int $(0,0)$
$(-2,0)$
$(2,0)$

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{f(4) - f(2)}{4-2}$$
$$= \frac{\left[(4)^4 - 4(4)^2\right] - \left[(2)^4 - 4(2)^2\right]}{2}$$
$$= \frac{(256-64) - (16-16)}{2}$$
$$= \frac{192}{2} = 96$$

← Slope of the line between $x=2$ & $x=4$



A secant (chord) is a line that crosses a curve at two or more points.



Instantaneous Rate of Change: The amount of change of a function at a particular value.

Ex: the value that the speedometer reads.

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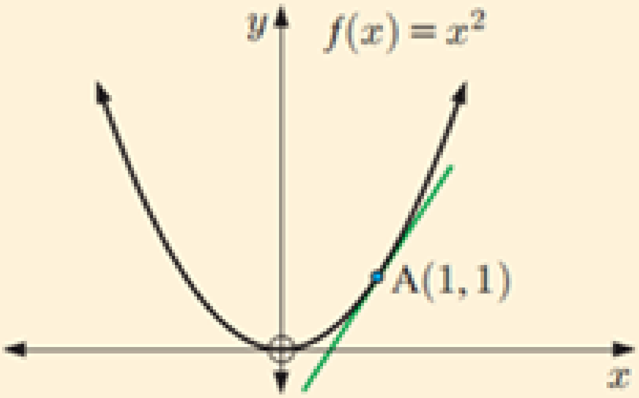
The Gradient of a Tangent

INVESTIGATION 3 — THE GRADIENT OF A TANGENT



Given a curve $f(x)$, we wish to find the gradient of the tangent at the point $(a, f(a))$.

In this Investigation we will find the gradient of the tangent to $f(x) = x^2$ at the point A(1, 1).

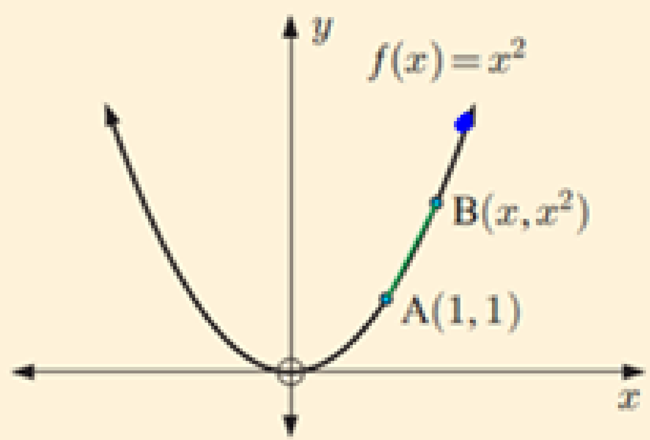
DEMO

• (5, 25)

What to do:

1 Suppose B lies on $f(x) = x^2$, and B has coordinates (x, x^2).

a Show that the chord [AB] has gradient $\dfrac{x^2 - 1}{x - 1}$. = Slope = $\dfrac{\Delta y}{\Delta x}$



b Copy and complete the table alongside:

c Comment on the gradient of [AB] as x gets closer to 1.

2 Repeat the process letting x get closer to 1, but from the left of A. Use the points where $x = 0, 0.8, 0.9, 0.99$, and 0.999.

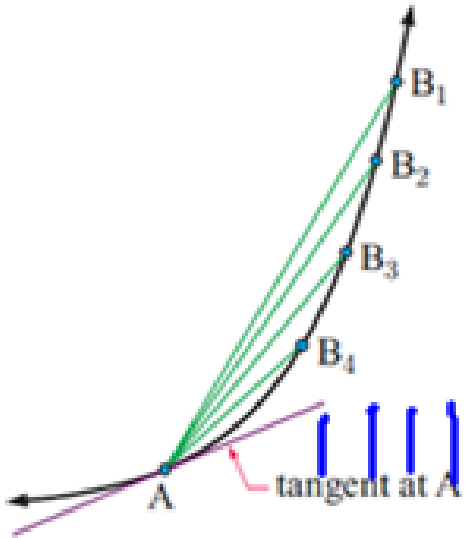
3 Click on the icon to view a demonstration of the process.

4 What do you suspect is the gradient of the tangent at A?

x	Point B	gradient of [AB]
5	(5, 25)	6 $\frac{25-1}{5-1} =$
3	(3,9)	4
2	(2,4)	3
1.5	(1.5, 2.25)	2.5
1.1	(1.1, 1.21)	2.1
1.01		2.01
1.001		2.001

$\text{slope} = \frac{\Delta y}{\Delta x}$

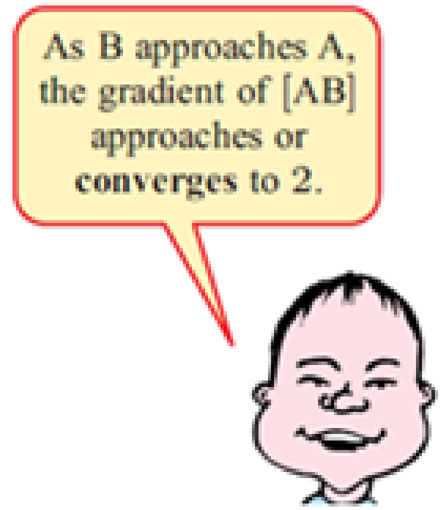
From **Investigation 3**, the gradient of [AB] $= \dfrac{x^2-1}{x-1}$.



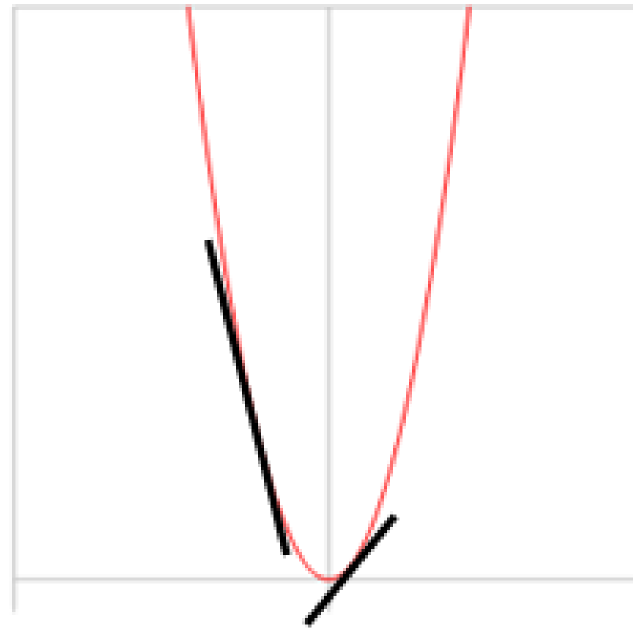
As B approaches A, $x \to 1$ and
the gradient of [AB] $\to$ the gradient of the tangent at A.

So, the gradient of the tangent at the point A is

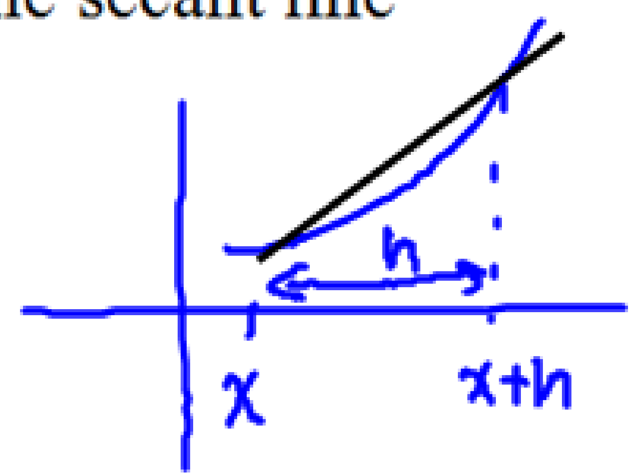
$$\begin{aligned} m_T &= \lim_{x\to 1} \frac{x^2-1}{x-1} \\ &= \lim_{x\to 1} \frac{(x+1)(x-1)}{x-1} \\ &= \lim_{x\to 1} (x+1) \quad \text{since } x \neq 1 \\ &= 2 \end{aligned}$$



A tangent is a line that touches a curve in exactly one point, or touches a curve but does not cross it. The tangent has the same steepness as the curve at that point



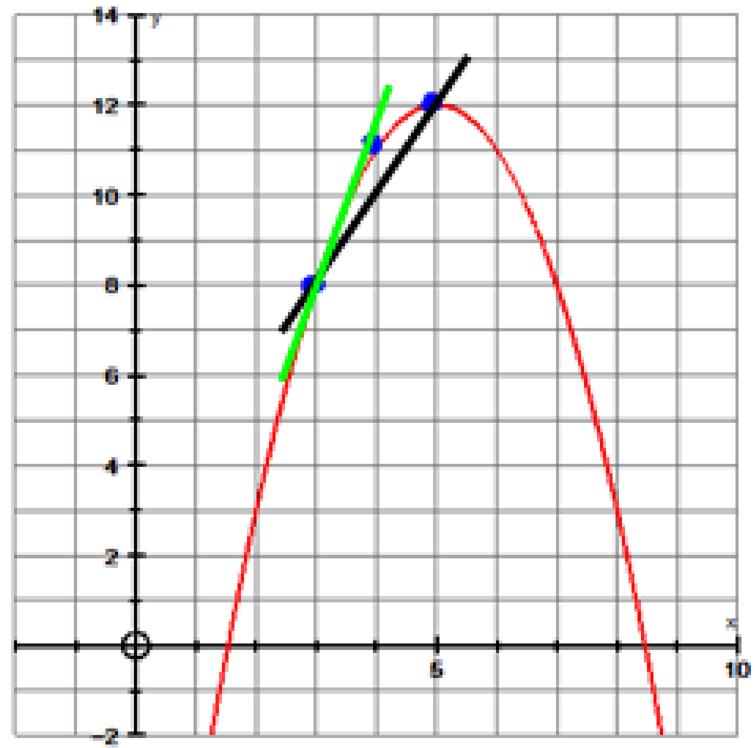
- When the distance between any two points on a graph becomes smaller, the secant becomes more like a tangent than a secant. As *h* **approaches** zero, the secant line (chord) **approaches** the tangent line.



Example: Given the function

$f(x) = -x^2 + 10x - 13$

Find the average rate of change between



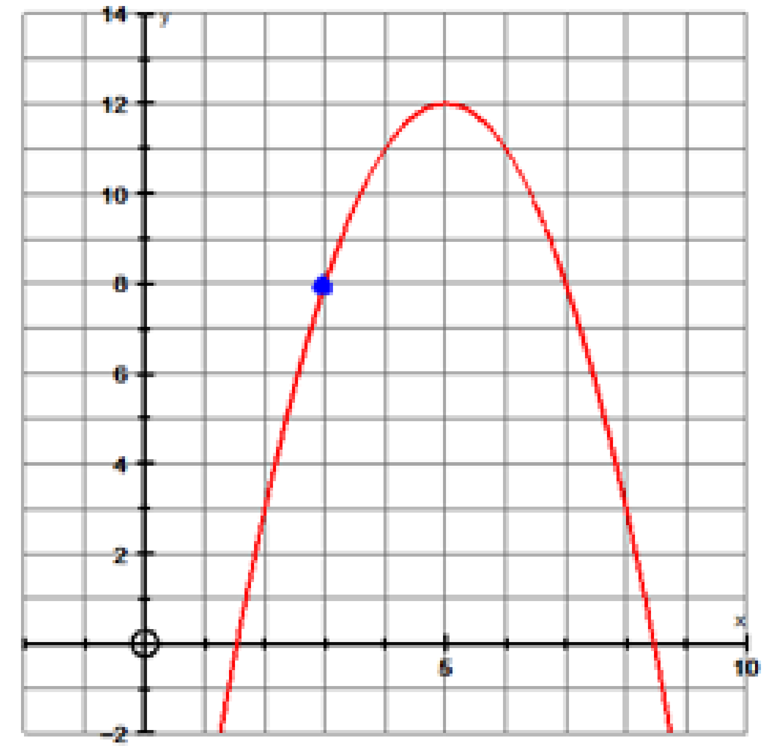
A) (3,8) and (5,12)

$m_{secant} = \frac{\Delta y}{\Delta x} = \frac{12-8}{5-3} = \frac{4}{2} = 2$

B) (3,8) and (4, 11)

$m = \frac{\Delta y}{\Delta x} = \frac{11-8}{4-3} = \frac{3}{1}$

C) Find the instantaneous rate of change at x=3



$$f(x) = -x^2 + 10x - 13$$

$$m_{tangent} = \lim_{h \to 0} m_{secant}$$

$$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{(x+h) - x}$$

$$= \lim_{h \to 0} \frac{f(3+h) - f(3)}{h}$$

$$= \lim_{h \to 0} \frac{\left[-(3+h)^2 + 10(3+h) - 13\right] - \left[-(3)^2 + 10(3) + 3\right]}{h}$$

$$= \lim_{h\to 0} \frac{\left[-(3+h)^2+10(3+h)-13\right]-\left[-(3)^2+10(3)-13\right]}{h}$$

$$= \lim_{h\to 0} \frac{\left(-(9+6h+h^2)+30+10h-13\right)-\left(-9+30-13\right)}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{-9}-6h-h^2\cancel{+30}+10h\cancel{-13}\cancel{+9}\cancel{-30}\cancel{+13}}{h}$$

$$= \lim_{h\to 0} \frac{-h^2+4h}{h} = \lim_{h\to 0} \frac{\cancel{h}(-h+4)}{\cancel{h}}$$

$$= \lim_{h\to 0} -h+4 = 4$$

HW: find the instantaneous rate of change of $f(x) = x^2+4x$

at $x=0$, $x=-2$, $x=-4$

$\uparrow \qquad \uparrow \qquad \uparrow$

$4 \qquad 0 \qquad -4$