

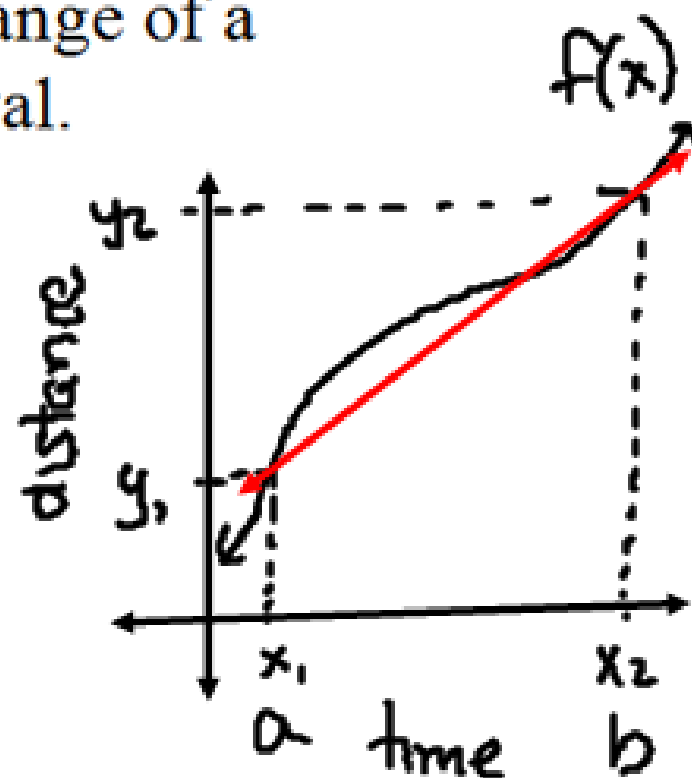
Ch 14C - Rates of Change

Average rate of change: The amount of change of a function divided by the length of the interval.

Ex: Average speed = $\frac{\text{how far you went}}{\text{how long it took}}$

$$\text{slope} = m = \frac{\Delta y}{\Delta x} = \frac{f(b) - f(a)}{b - a}$$

(gradient)



Example: Find the average rate of change of $f(x) = x^4 - 4x^2$ over the interval $[2,4]$

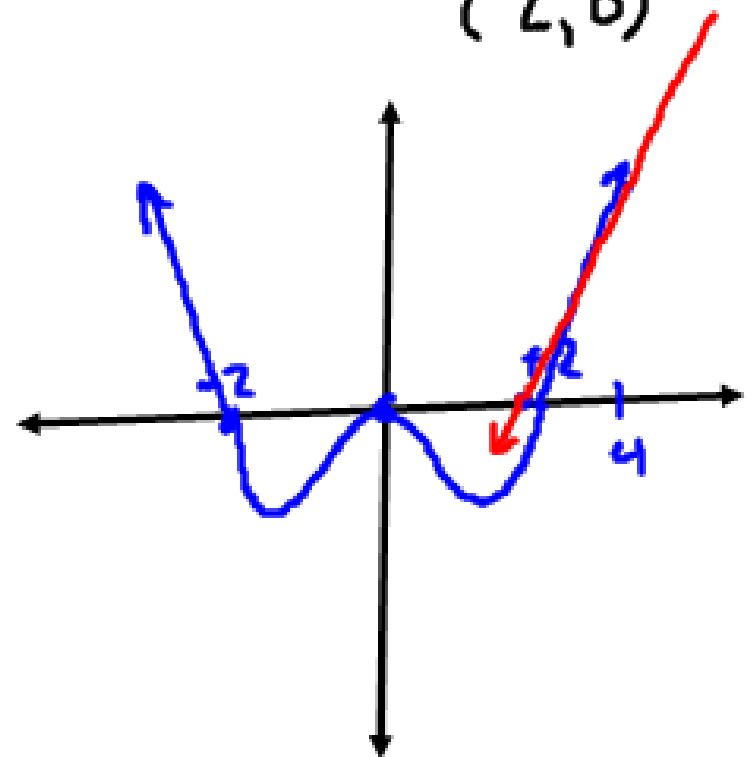
$$x=2, x=4$$

$$\begin{aligned} m &= \frac{f(4) - f(2)}{4 - 2} \\ &= \frac{[(4)^4 - 4(4)^2] - [(2)^4 - 4(2)^2]}{2} \\ &= \frac{(192) - 0}{2} \\ &= 96 \end{aligned}$$

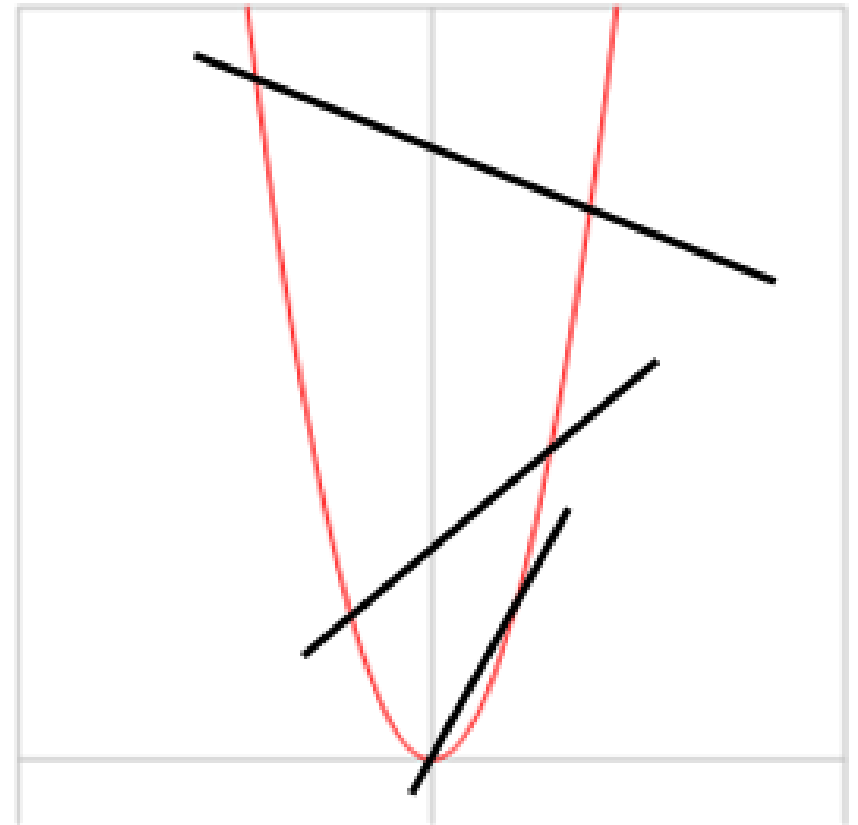
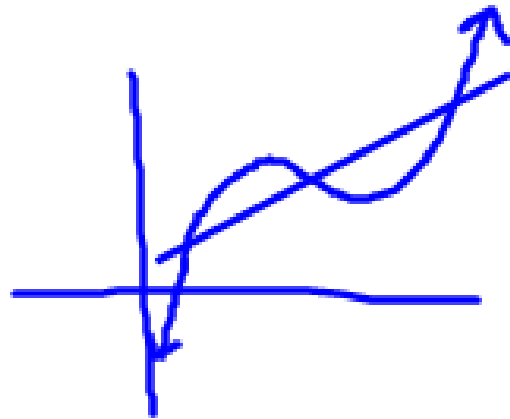
$$= x^2(x^2 - 4)$$

$$0 = x^2(x+2)(x-2)$$

$$\begin{aligned} x\text{-int: } &(0,0) \\ &(-2,0) \\ &(2,0) \end{aligned}$$



A secant (chord) is a line that crosses a curve at two or more points.



Instantaneous Rate of Change: The amount of change of a function at a particular value.

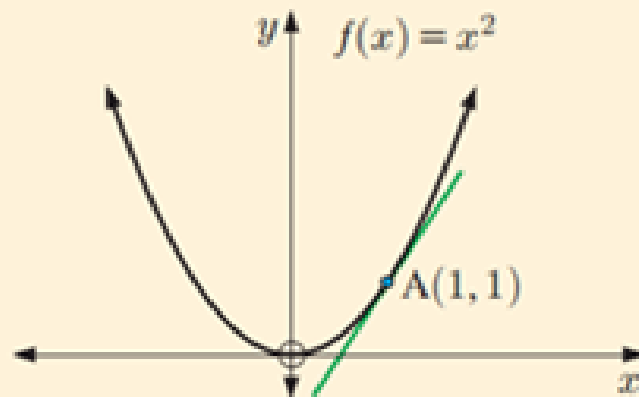
Ex: the value that the speedometer reads.

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The Gradient of a Tangent

INVESTIGATION 3

THE GRADIENT OF A TANGENT



Given a curve $f(x)$, we wish to find the gradient of the tangent at the point $(a, f(a))$.

In this Investigation we will find the gradient of the tangent to $f(x) = x^2$ at the point $A(1, 1)$.

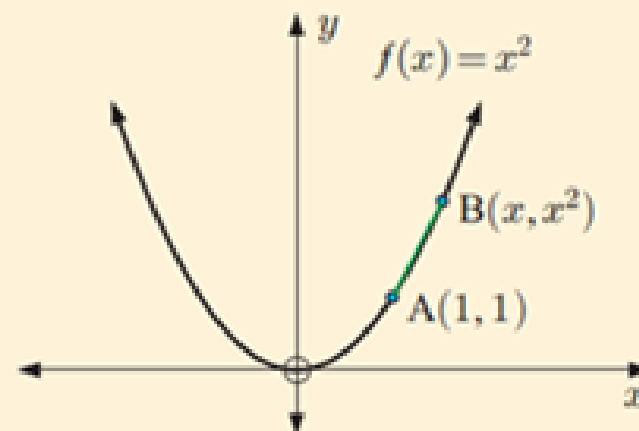


What to do:

1 Suppose B lies on $f(x) = x^2$, and B has coordinates (x, x^2) .

a Show that the chord $[AB]$ has gradient

$$\frac{x^2 - 1}{x - 1} = \text{Dlope}$$
$$= \frac{(x-1)(x+1)}{(x-1)} = x+1$$



b Copy and complete the table alongside: $(1, 1)$

c Comment on the gradient of [AB] as x gets closer to 1.

2 Repeat the process letting x get closer to 1, but from the left of A. Use the points where $x = 0, 0.8, 0.9, 0.99,$ and 0.999 .

3 Click on the icon to view a demonstration of the process.

4 What do you suspect is the gradient of the tangent at A?

x	Point B	gradient of [AB]
5	(5, 25)	6
3	(3, 9)	4
2	(2, 4)	3
1.5	(1.5, 2.25)	2.5
1.1		2.1
1.01		2.01
1.001		2.001

0

0.8

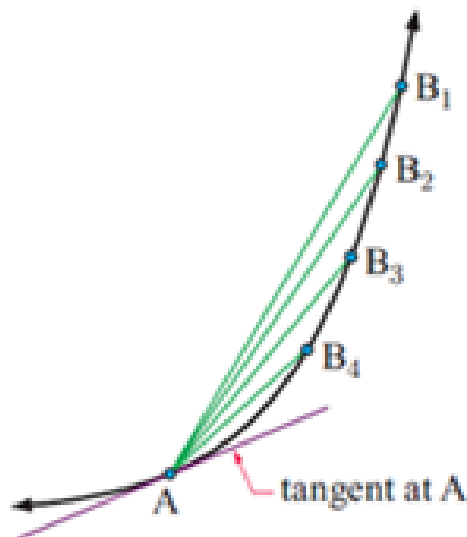
0.9

1

1.8

1.9

From **Investigation 3**, the gradient of $[AB] = \frac{x^2 - 1}{x - 1}$.

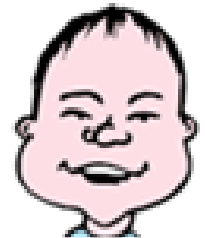


As B approaches A, $x \rightarrow 1$ and the gradient of $[AB] \rightarrow$ the gradient of the tangent at A.

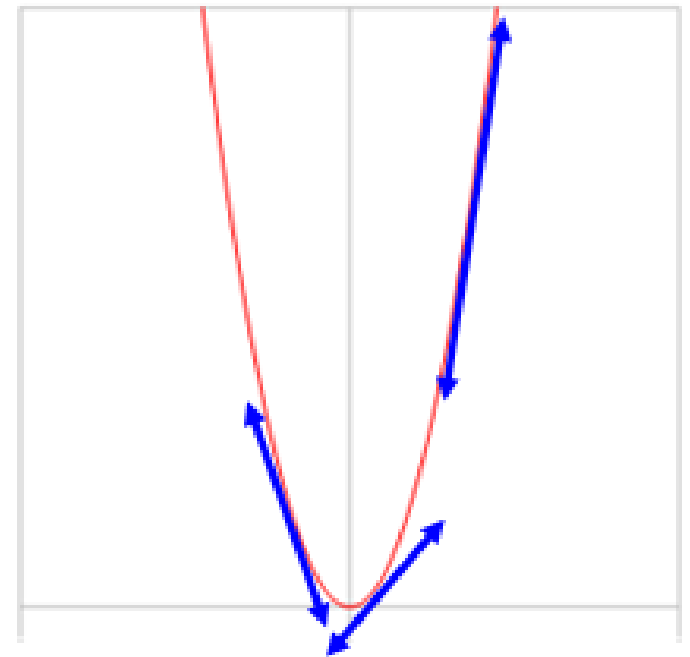
So, the gradient of the tangent at the point A is

$$\begin{aligned} m_T &= \lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} \\ &= \lim_{x \rightarrow 1} \frac{(x + 1)(x - 1)}{x - 1} \\ &= \lim_{x \rightarrow 1} (x + 1) \quad \text{since } x \neq 1 \\ &= 2 \end{aligned}$$

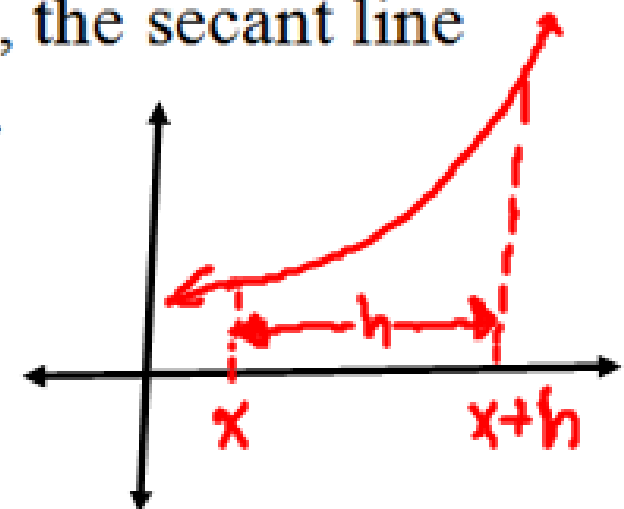
As B approaches A, the gradient of $[AB]$ approaches or converges to 2.



A tangent is a line that touches a curve in exactly one point, or touches a curve but does not cross it. The tangent has the same steepness as the curve at that point



- When the distance between any two points on a graph becomes smaller, the secant becomes more like a tangent than a secant. As h **approaches** zero, the secant line (chord) **approaches** the tangent line.



Example: Given the function

$$f(x) = -x^2 + 10x - 13$$

Find the average rate of change between

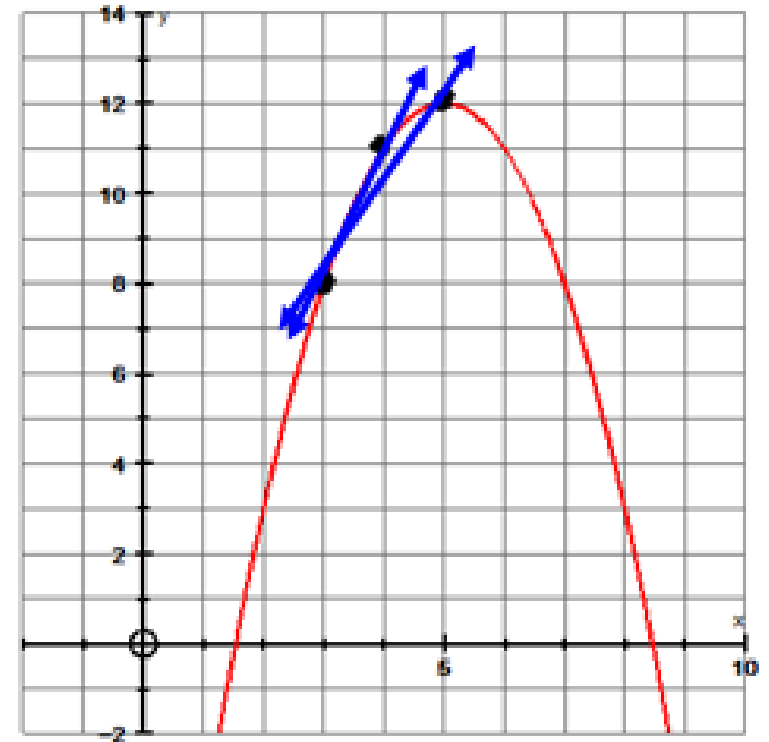
Slope of secant line

A) (3,8) and (5,12)

$$m = \frac{\Delta y}{\Delta x} = \frac{12-8}{5-3} = \frac{4}{2} = 2$$

B) (3,8) and (4,11)

$$m = \frac{\Delta y}{\Delta x} = \frac{11-8}{4-3} = 3$$



C) Find the instantaneous rate of change at $x=3$

- the second x value gets really close to $x=3$



$$m_{\text{tangent}} = \lim_{h \rightarrow 0} m_{\text{secant}} = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{(x+h) - x}$$
$$= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$M_{\text{tangent}} = \lim_{h \rightarrow 0} \frac{f(3+h) - f(3)}{h}$$

$$f(x) = -x^2 + 10x - 13$$

$$= \lim_{h \rightarrow 0} \frac{[-(3+h)^2 + 10(3+h) - 13] - [-3^2 + 10(3) - 13]}{h}$$

$$= \lim_{h \rightarrow 0} \frac{-(9+6h+h^2) + 30 + 10h - 13 - [-9 + 30 - 13]}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\cancel{-9} - 6h - h^2 + \cancel{30} + 10h - \cancel{13} + \cancel{9} - \cancel{30} + \cancel{13}}{h}$$

$$= \lim_{h \rightarrow 0} \frac{4h - h^2}{h} = \lim_{h \rightarrow 0} \frac{h(4-h)}{h} = \lim_{h \rightarrow 0} 4-h = 4$$

HW: Find the instantaneous rate of change of $f(x) = x^2 + 4x$

at $x=0$, $x=-2$, $x=-4$

\uparrow	\uparrow	\uparrow
4	0	-4