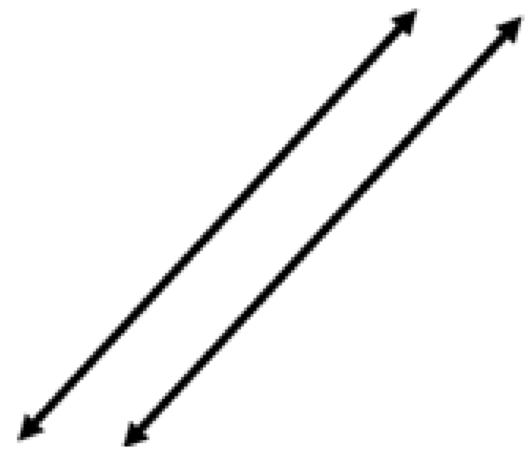
13G – Relationships Between Lines

Two lines in space can be classified as:

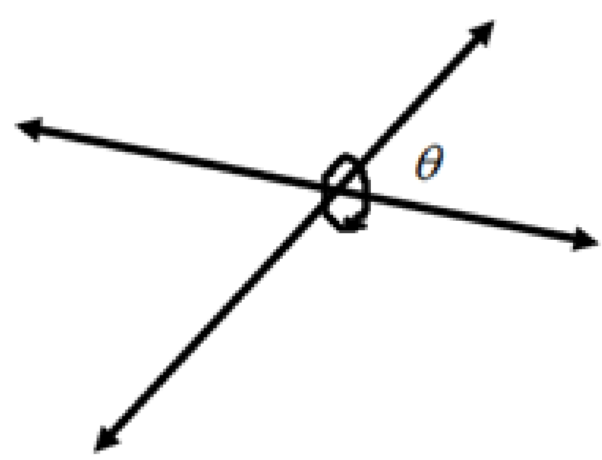
Parallel

The angle between them is 0°.
(lines do not meet – no solutions)



Intersecting

The angle between them is $\theta°$.
(one point of intersection, unique solution)



Coincident

The same line.
(infinitely many solutions)

$y = 3x + 7$
$2y = 6x + 14$

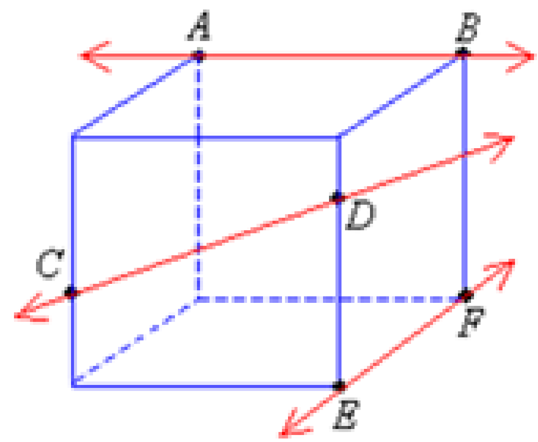
Lines are **coplanar** if they lie in the same plane.
If lines are not coplanar then they are **skew**.

If the lines are coplanar, they may be intersecting, parallel, or coincident.

Skew

Lines are neither parallel nor intersecting.

AB and EF are in different planes – will never intersect.

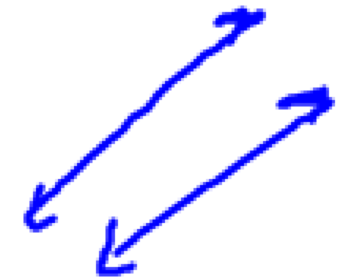


If the lines are skew, there is still an angle one line makes with the other.

Translate one line to intersect the other, then the angle between the intersecting lines is defined as the angle between the original lines.

Examples:

1. Line 1 has equations $x = 3 - 2t$, $y = 1 + t$, and $z = -2 + 3t$.
Line 2 has equations $x = 6s$, $y = 4 - 3s$, and $z = -3 - 9s$.
Show the lines are parallel.



L_1: $\vec{r} = \vec{a} + t\vec{b}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} + t\begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}$$

$\Rightarrow$ direction vector $\vec{b_1} = \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}$

$\vec{b_2} = -3\vec{b_1}$

L_2:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \\ -3 \end{pmatrix} + s\begin{pmatrix} 6 \\ -3 \\ -9 \end{pmatrix}$$

$\Rightarrow$ Direction vector $\vec{b_2} = \begin{pmatrix} 6 \\ -3 \\ -9 \end{pmatrix} = -3\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$

→ show that they are no coincident

L_1: let $t=0$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$$

If the lines are coincident, then $\begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$ must be on L_2

$x=3$

$y=1$

$z=-2$

L_2:

$$\left.\begin{aligned} x &= 6s \\ 3 &= 6s \\ \tfrac{1}{2} &= s \end{aligned}\right\}$$

$y = 4-3s$

$1 = 4-3s$

$-3 = -3s$

$1 = s$

2. Show the following lines intersect and find the angle between them.

both lines will share (x,y,z)

Line 1: $x = 1 - r$ $y = r$ $z = 3 - 2r$

Line 2: $x = 1 + 2t$ $y = -1 - t$ $z = 4 + 3t$

$1 - r = 1 + 2t$ $r = -2t$	$r = -1 - t$	$3 - 2r = 4 + 3t$ $3 - 2(-2) \stackrel{?}{=} 4 + 3(1)$ $3 + 4 \stackrel{?}{=} 4 + 3$ ✓ They will intersect

Substitution

$$\left.\begin{aligned} -2t &= -1 - t \\ 1 &= t \\ \therefore r &= -2(1) \\ r &= -2 \end{aligned}\right\} \text{plug into } z$$

Line 1: $x = 1 - r$ $\quad y = r$ $\quad z = 3 - 2r$ $\rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} + r\begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix}$

Line 2: $x = 1 + 2t$ $\quad y = -1 - t$ $\quad z = 4 + 3t$ $\rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} + t\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$

$$\cos\theta = \frac{|\vec{b_1} \cdot \vec{b_2}|}{|\vec{b_1}| \cdot |\vec{b_2}|}$$

$$= \frac{|(-1)(2) + (1)(-1) + (-2)(3)|}{\left(\sqrt{(-1)^2 + (1)^2 + (-2)^2}\right)\left(\sqrt{(2)^2 + (-1)^2 + (3)^2}\right)}$$

$$= \frac{|-2 - 1 - 6|}{\sqrt{6} \cdot \sqrt{14}}$$

$$\cos\theta = \frac{9}{\sqrt{6 \cdot 14}}$$

$$\cos\theta = 0.98198$$

$$\theta = 10.89^\circ$$

3. Show that the following lines are skew.

Line 1: $x = -1 + 2r \quad y = 1 - 2r \quad z = 1 + 4r$

$\vec{b_1} = \begin{pmatrix} 2 \\ -2 \\ 4 \end{pmatrix}$

Line 2: $x = 1 + 2t \quad y = -1 - t \quad z = 4 + 3t$

$\vec{b_2} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$

$-1 + 2r = 1 + 2t$

$2 = 2r - 2t$ ①

$1 - 2r = -1 - t$

$2 = 2r - t$ ②

$1 + 4r = 4 + 3t$

$1 + 4(1) \stackrel{?}{=} 4 + 3(0)$

$5 \neq 4$ X

$\therefore$ the lines do not intersect

elimination

$$\begin{array}{rl} & -2 = -2r + 2t \\ + & 2 = 2r - t \\ \hline & 0 = 0 + t \\ & 0 = t \end{array}$$

$\therefore 2 = 2r \Rightarrow r = 1$

now we need to check that they are not parallel

$$\begin{pmatrix} 2 \\ -2 \\ 4 \end{pmatrix} \neq k \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$$

$$-2 = -k$$
$$2 = k$$

$$2 = 2k$$
$$1 = k$$

they are not parallel!! :)