13D – Constant Velocity Problems

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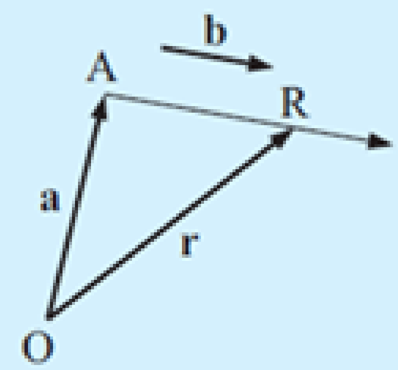
If an object has initial position vector **a** and moves with constant velocity **b**, its position at time t is given by

$$\mathbf{r} = \mathbf{a} + t\mathbf{b} \quad \text{for} \quad t \geqslant 0.$$

The **speed** of the object is $|\mathbf{b}|$.



Example: $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 6 \\ -4 \end{pmatrix} + t\begin{pmatrix} 3 \\ 7 \end{pmatrix}$ is the vector equation of the path of an object where $t \geq 0$, t is in seconds. Distance units are metres.

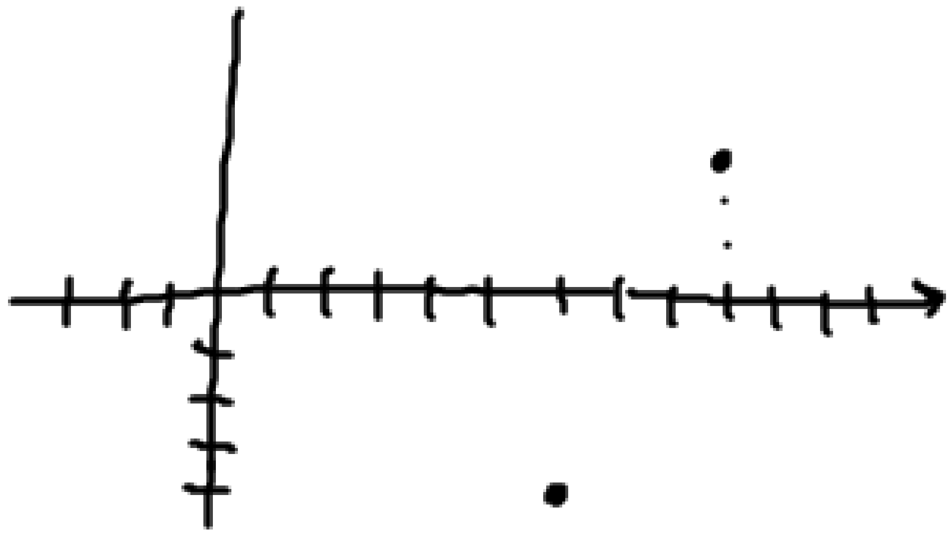
(a) Find the object's initial position. $t=0$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 6 \\ -4 \end{pmatrix}$$

(b) Plot the path of the object for $t = 0, 1, 2$.

$\underline{t=1}$ $\quad$ $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 9 \\ 3 \end{pmatrix}$

$\underline{t=2}$ $\quad$ $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 12 \\ 10 \end{pmatrix}$



(c) Find the velocity vector of the object.

$$\begin{pmatrix} 3 \\ 7 \end{pmatrix}$$

(d) Find the object's speed.

$$\text{speed} = \sqrt{(3)^2 + (7)^2} = \sqrt{9+49}$$
$$= \sqrt{58} \text{ m/s}$$

Example: An object is initially at (-3, 8) and moves with the velocity vector 2**i** + **j**

Find:

(a) the position of the object at any time *t*

$$\vec{r} = \vec{a} + t\vec{b}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 \\ 8 \end{pmatrix} + t\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

(b) the speed of the object

$$\text{speed} = \sqrt{2^2 + 1^2}$$

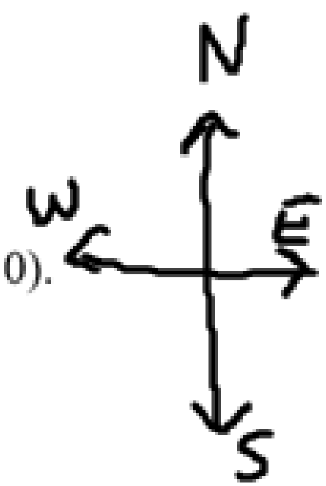
$$= \sqrt{5}$$

(c) the position at *t* = 3

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 \\ 8 \end{pmatrix} + 3\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 11 \end{pmatrix}$$

(d) the time when the object is due North of (0, 0).



$$x = 0$$

$$x = -3 + 2t$$

$$0 = -3 + 2t$$

$$\frac{3}{2} = t$$

13E – The Shortest Distance from a Line to a Point

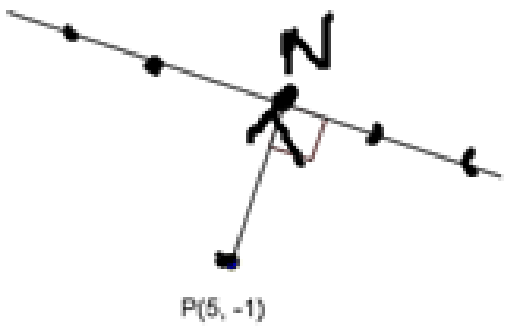
The shortest distance from a point to a line is the perpendicular distance.

direction vector

Example: A line has vector equation $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + t\begin{pmatrix} 3 \\ -1 \end{pmatrix}, \quad t \in R.$

Let P be the point (5, -1). Find exactly the shortest distance from P to the line.

let N Be the closest point on the line to P



N has coord pt $(1+3t, 2-t)$

vector $\overrightarrow{PN} = \begin{pmatrix} (1+3t)-5 \\ (2-t)-(-1) \end{pmatrix} = \begin{pmatrix} 3t-4 \\ 3-t \end{pmatrix}$

We know $\overrightarrow{PN} \cdot \begin{pmatrix} 3 \\ -1 \end{pmatrix} = 0$ if they are perpendicular

$$\begin{pmatrix} 3t-4 \\ 3-t \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -1 \end{pmatrix} = 0$$

$$(3t-4)(3) + (3-t)(-1) = 0$$

$$9t - 12 + t - 3 = 0$$

$$10t - 15 = 0$$

$$10t = 15$$

$$t = \frac{15}{10} = \frac{3}{2}$$

We know

$$\overrightarrow{PN} = \begin{pmatrix} 3t-4 \\ 3-t \end{pmatrix} = \begin{pmatrix} 3(\frac{3}{2})-4 \\ 3-\frac{3}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{3}{2} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$|\overrightarrow{PN}| = \frac{1}{2}\sqrt{(1)^2 + (3)^2} = \frac{1}{2}\sqrt{10}$$

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direction vector

2 An ocean liner is at $(6, -6)$, cruising at $10\ \mathrm{km\,h^{-1}}$ in the direction $\begin{pmatrix} -3 \\ 4 \end{pmatrix}$.

A fishing boat is anchored at $(0, 0)$. Distances are in kilometres.

- **a** Find, in terms of $\mathbf{i}$ and $\mathbf{j}$, the initial position vector of the liner from the fishing boat.
- **b** Write an expression for the position vector of the liner at any time t hours after it has sailed from $(6, -6)$.
- **c** When will the liner be due east of the fishing boat?
- **d** Find the time and position of the liner when it is nearest to the fishing boat.

A) $6\vec{i} - 6\vec{j}$

velocity vector

B) $\vec{r} = \vec{a} + t\vec{b}$

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 6 \\ -6 \end{pmatrix} + t\begin{pmatrix} -6 \\ 8 \end{pmatrix}$

length of direction vector $= \sqrt{(-3)^2 + 4^2}$

$= \sqrt{25}$

$= 5$

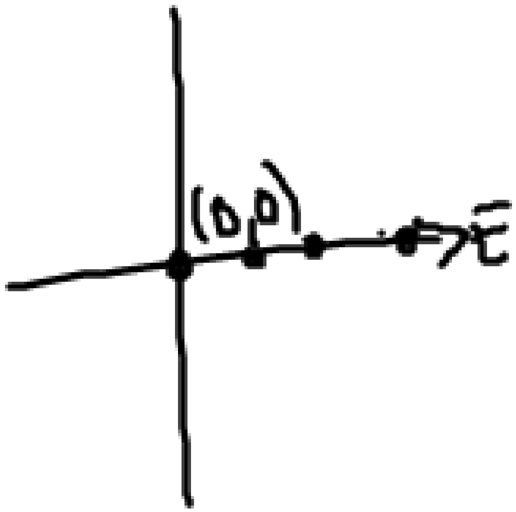
*the velocity is 2 times the length of the direction vector

c) Due east $y=0$ par. $y=-6+8t$

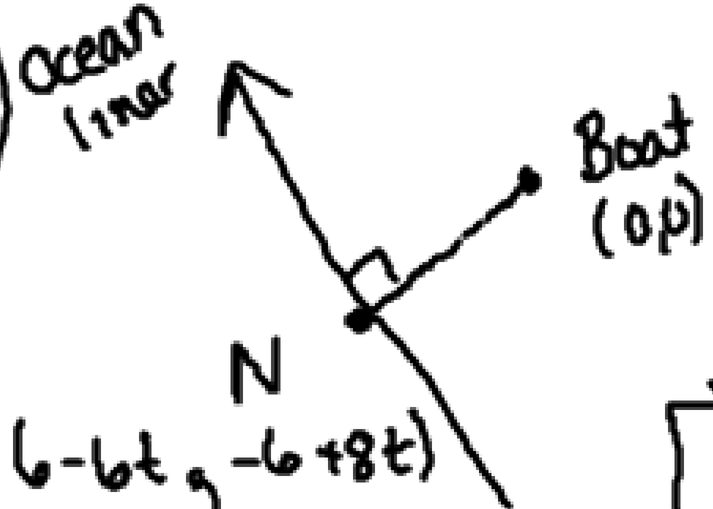
$0=-6+8t$

$8t=6$

$t=\frac{3}{4}$ hr



D)



$(6-6t, -6+8t)$

$$\overrightarrow{BN} = \begin{pmatrix} 6-6t \\ -6+8t \end{pmatrix}$$

$$\overrightarrow{BN} \cdot \begin{pmatrix} -3 \\ 4 \end{pmatrix} = 0$$

← direction vector

$$\begin{pmatrix} 6-6t \\ -8+8t \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 4 \end{pmatrix} = 0$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 6 \\ -6 \end{pmatrix} + t \begin{pmatrix} -6 \\ 8 \end{pmatrix}$$

$$\to (6-6t)(-3) + (-8+8t)(4) = 0$$

$$-18 + 18t - 32 + 32t = 0$$

$$50t - 42 = 0$$

$$50t = 42$$

$$t = \frac{42}{50}$$

$$N\left(6 - 6\left(\frac{42}{50}\right), -6 + 8\left(\frac{42}{50}\right)\right)$$

$$N\,(0{,}96, 0{,}72)$$

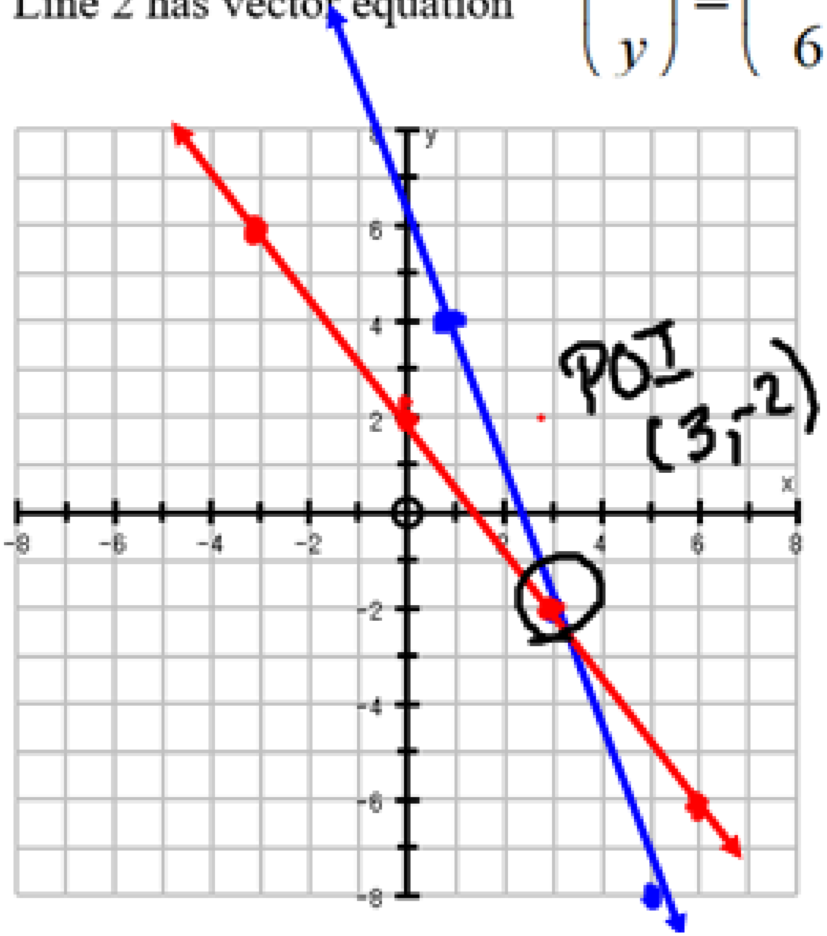
13F – Intersecting Lines

Vector equations of two lines can be solved to find the point of intersection.

Example:

Line 1 has vector equation $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} + s\begin{pmatrix} 2 \\ -6 \end{pmatrix}$

Line 2 has vector equation $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 \\ 6 \end{pmatrix} + t\begin{pmatrix} 3 \\ -4 \end{pmatrix}$ (*s* and *t* are scalars)



The two lines meet at E.

Use vector methods to find the coordinates of E.

$$\begin{pmatrix}1\\4\end{pmatrix} + s\begin{pmatrix}2\\-6\end{pmatrix} = \begin{pmatrix}-3\\6\end{pmatrix} + t\begin{pmatrix}3\\-4\end{pmatrix}$$

Paraform

sub/elim

mult. eq^n ① by 3

$1+2s = -3+3t \Rightarrow 4 = -2s+3t$ ①

$4-6s = 6-4t \Rightarrow -2 = 6s-4t$ ②

$$\begin{array}{rrl} & 12 = & -6s+9t \\ + & -2 = & 6s-4t \\ \hline & 10 = & 5t \end{array}$$

$2 = t$

$$\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}-3\\6\end{pmatrix} + t\begin{pmatrix}3\\-4\end{pmatrix}$$

$$\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}-3\\6\end{pmatrix} + 2\begin{pmatrix}3\\-4\end{pmatrix}$$

$$= \begin{pmatrix}3\\-2\end{pmatrix}$$

$s = 1$