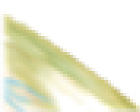
Chapter 13

Vector applications

Syllabus reference: 4.1, 4.2, 4.3, 4.4

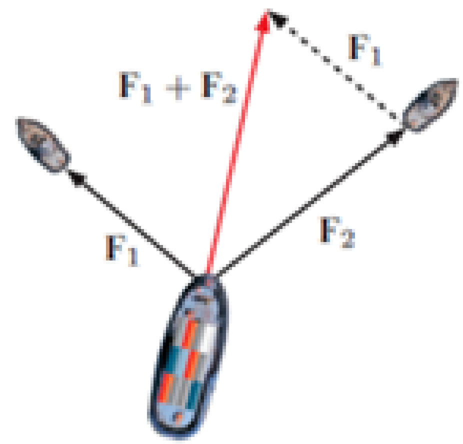
Contents:



A PROBLEMS INVOLVING VECTOR OPERATIONS

When we apply vectors to problems in the real world, we often consider the combined effect when vectors are added together. This sum is called the **resultant vector**.

We have an example of vector addition when two tug boats are used to pull a ship into port. If the tugs tow with forces $\mathbf{F}_1$ and $\mathbf{F}_2$ then the resultant force is $\mathbf{F}_1 + \mathbf{F}_2$.



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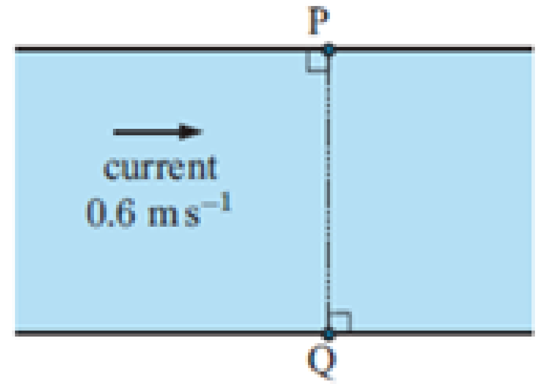
2 In still water, Mary can swim at $1.2\ \text{m s}^{-1}$. She is standing at point P on the edge of a canal, directly opposite point Q. The water is flowing to the right at a constant speed of $0.6\ \text{m s}^{-1}$.



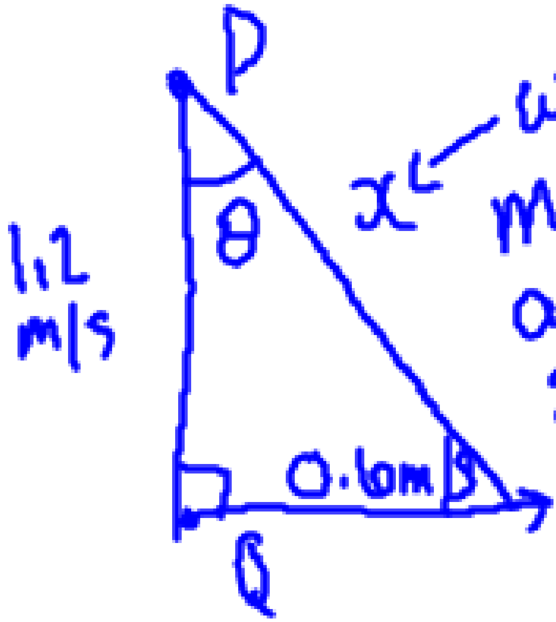
a If Mary tries to swim directly from P to Q without allowing for the current, what will her actual velocity be?

b Mary wants to swim directly across the canal to point Q.

i At what angle should she *aim* to swim in order that the current corrects her direction?

ii What will Mary's actual speed be?

A)



← what Mary's actually swimming

Pythag: $a^2+b^2=c^2$

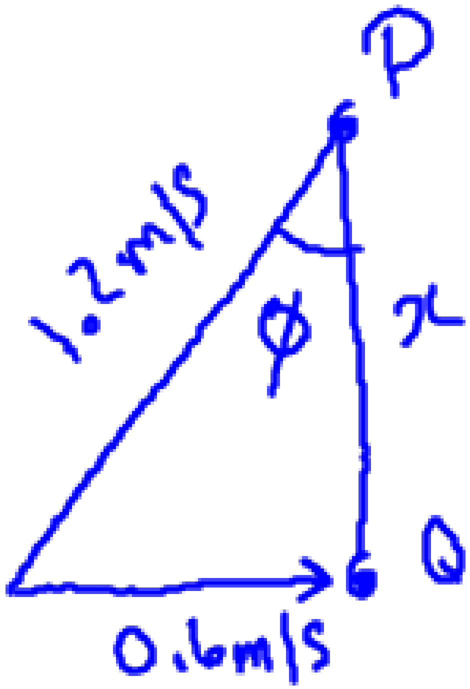
$(1.2)^2+(0.6)^2=x^2$

$1.8=x^2$

$1.34\text{m/s}=x$

$\tan\theta=\frac{0.6}{1.2}$

$\theta\sim26.6^\circ$



$$x^2 + 0.6^2 = 1.2^2$$

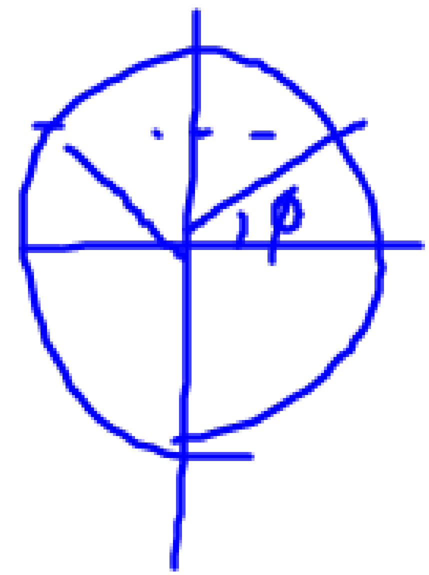
$$x^2 = 1.08$$

$$x = 1.039 \text{ m/s}$$

$$\sin\phi = \frac{0.6}{1.2}$$

$$\sin\phi = \frac{1}{2}$$

$$\phi = 30^\circ$$

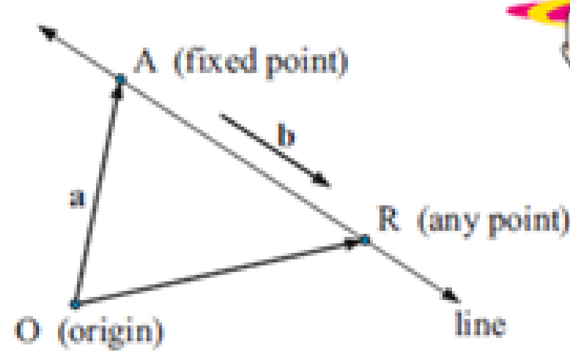


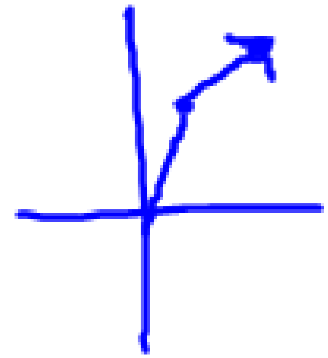
B LINES IN 2-D AND 3-D

In both 2-D and 3-D geometry we can determine the **equation of a line** using its **direction** and any **fixed point** on the line.

Suppose a line passes through a fixed point A with position vector **a**, and that the line is parallel to the vector **b**.

DEMO





Consider a point R on the line so that $\overrightarrow{OR} = \mathbf{r}$.

By vector addition, $\overrightarrow{OR} = \overrightarrow{OA} + \overrightarrow{AR}$

$$\therefore \quad \mathbf{r} = \mathbf{a} + \overrightarrow{AR}.$$

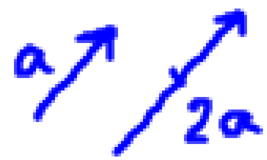
Since $\overrightarrow{AR}$ is parallel to **b**,

$$\overrightarrow{AR} = t\mathbf{b} \quad \text{for some scalar } t \in \mathbb{R}$$

$$\therefore \quad \mathbf{r} = \mathbf{a} + t\mathbf{b}$$

So, $\boxed{\mathbf{r} = \mathbf{a} + t\mathbf{b}, \quad t \in \mathbb{R} \quad \text{is the vector equation of the line.}}$

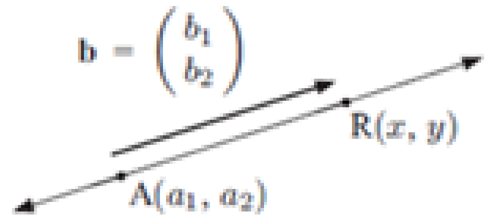
$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + t\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$



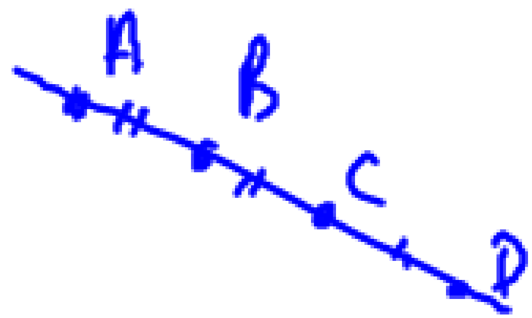
LINES IN 2-D

- In 2-D we are dealing with a **line in a plane.**
- $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + t\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ is the **vector equation** of the line

  where R(x, y) is any point on the line,
  A(a_1, a_2) is a known fixed point on the line,
  and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ is the **direction vector** of the line.



- The gradient of the line is $m = \dfrac{b_2}{b_1}$.
- Since $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_1 + b_1 t \\ a_2 + b_2 t \end{pmatrix}$, we can write the **parametric equations** of the line $x = a_1 + b_1 t$ and $y = a_2 + b_2 t$, where $t \in \mathbb{R}$ is the **parameter.**
  Each point on the line corresponds to exactly one value of t.



- We can convert these equations into Cartesian form by equating t values.
  Using $t = \dfrac{x - a_1}{b_1} = \dfrac{y - a_2}{b_2}$ we obtain $b_2 x - b_1 y = b_2 a_1 - b_1 a_2$
  which is the **Cartesian equation** of the line.

EXERCISE 13B

1 Describe each of the following lines using:

i a vector equation   ii parametric equations   iii a Cartesian equation

b a line passing through (5, 2) which is perpendicular to $\begin{pmatrix} 5 \\ 2 \end{pmatrix} = \vec{p}$

direction vector

$\vec{p} \cdot \vec{b} = 0$

$\begin{pmatrix} 5 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = 0$

$5b_1 + 2b_2 = 0$

direction vector $\begin{pmatrix} -2 \\ 5 \end{pmatrix}$

vector eqn

$\vec{r} = \vec{a} + t\vec{b}$

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \end{pmatrix} + t\begin{pmatrix} -2 \\ 5 \end{pmatrix}$

parametric

$x = 5 - 2t$

$y = 2 + 5t$

cartesian

$x = 5 - 2t \qquad y = 2 + 5t$

$\frac{-2t}{-2} = \frac{x-5}{-2} \qquad y - 2 = 5t$

$t = \frac{x-5}{-2} \qquad t = \frac{y-2}{5}$

$$\frac{x-5}{-2} = \frac{y-2}{5}$$

$$5x - 25 = -2y + 4$$

$$5x + 2y = 29$$

3 **a** Does $(3, -2)$ lie on the line with vector equation $\mathbf{r} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} + t\begin{pmatrix} 1 \\ -3 \end{pmatrix}$?

b $(k, 4)$ lies on the line with parametric equations $x = 1 - 2t$, $y = 1 + t$. Find k.

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} + t\begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

$$\begin{pmatrix} 3 \\ -2 \end{pmatrix} \stackrel{?}{=} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + t\begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

give the same value of t?

$3 = 2 + t$
$t = 1$

$-2 = 1 - 3t$
$-3 = -3t$
$1 = t$

Since the value of t is the same, we know $(3, -2)$ is on the line

3 **a** Does (3, −2) lie on the line with vector equation $\mathbf{r} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} + t\begin{pmatrix} 1 \\ -3 \end{pmatrix}$?

b $(k, 4)$ lies on the line with parametric equations $x = 1 - 2t$, $y = 1 + t$. Find k.

we can find t using y-coord pt.

$4 = 1 + t$

$3 = t$

$k = 1 - 2t$

$k = 1 - 2(3)$

$k = -5$

5 Describe each of the following lines using:

i a vector equation **ii** parametric equations

~~a~~ ~~a line parallel to~~ $\cancel{\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}}$ ~~which passes through $(1, 3, -7)$~~

b a line which passes through $(0, 1, 2)$ with direction vector $\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ $\rightarrow \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$

$$\vec{r} = \vec{a} + t\vec{b}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

parametric: $x = t$
$y = 1 + t$
$z = 2 - 2t$

6 Find the vector equation of the line which passes through:

a A(1, 2, 1) and B(−1, 3, 2)

$\Rightarrow$ need a direction vector

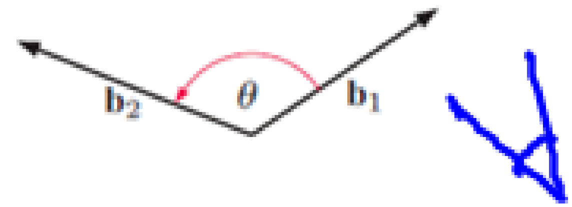
$$\vec{AB} = \begin{pmatrix} -1-1 \\ 3-2 \\ 2-1 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$$

$$\vec{r} = \vec{a} + t\vec{b}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + t\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$$

C THE ANGLE BETWEEN TWO LINES

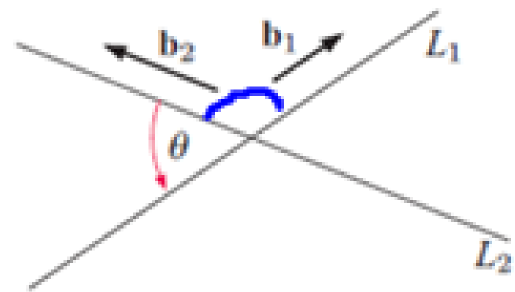
In **Chapter 12** we saw that the angle between two vectors is measured in the range $0^\circ \leqslant \theta \leqslant 180^\circ$. We used the formula $\cos\theta = \dfrac{\mathbf{b}_1 \bullet \mathbf{b}_2}{|\mathbf{b}_1||\mathbf{b}_2|}$.



In the case of lines which continue infinitely in both directions, we agree to talk about the *acute* angle between them. For an acute angle, $\cos\theta > 0$, so we use the formula

$$\boxed{\cos\theta = \frac{|\mathbf{b}_1 \bullet \mathbf{b}_2|}{|\mathbf{b}_1||\mathbf{b}_2|}}$$

always positive



where $\mathbf{b}_1$ and $\mathbf{b}_2$ are the direction vectors of the given lines L_1 and L_2 respectively.

EXERCISE 13C

1 Find the angle between the lines L_1: $x = -4 + 12t$, $y = 3 + 5t$
and L_2: $x = 3s$, $y = -6 - 4s$

Line 1

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -4 \\ 3 \end{pmatrix} + t\begin{pmatrix} 12 \\ 5 \end{pmatrix} \leftarrow b_1$$

direction vector

Line 2

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ -6 \end{pmatrix} + s\begin{pmatrix} 3 \\ -4 \end{pmatrix} \leftarrow b_2$$

direction vector

$$\cos\theta = \frac{|\vec{b}_1 \cdot \vec{b}_2|}{|\vec{b}_1| \cdot |\vec{b}_2|}$$

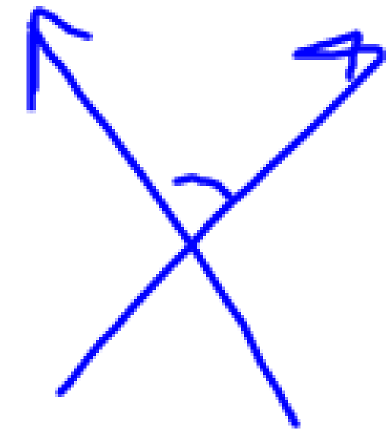
$$= \frac{|(12)(3) + (5)(-4)|}{\sqrt{12^2 + 5^2} \cdot \sqrt{(3)^2 + (-4)^2}}$$

$$\cos\theta = \frac{|36 - 20|}{\sqrt{169} \cdot \sqrt{25}}$$

$$\cos\theta = \frac{16}{13 \cdot 5} = 0.24615 \rightarrow \theta = 75.75^\circ$$

5 Find the measure of the angle between the lines:

a $x - y = 3$ and $3x + 2y = 11$



$y = x - 3$

slope $= \frac{1}{1} \left(\frac{\Delta y}{\Delta x}\right)$

direction vector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$2y = -3x + 11$

$y = \frac{-3}{2}x + \frac{11}{2}$

Slope $= \frac{-3}{2}$

has a slope of $\frac{-3}{2}$ but a direction vector of $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$.

$$\cos\theta = \frac{|(1)(2) + (1)(-3)|}{\sqrt{1^2 + 1^2} \cdot \sqrt{2^2 + (-3)^2}}$$

$$\cos\theta = \frac{|-1|}{\sqrt{2} \cdot \sqrt{13}}$$

$$\cos\theta = \frac{1}{\sqrt{26}}$$

$$\theta \approx 78.69^\circ$$