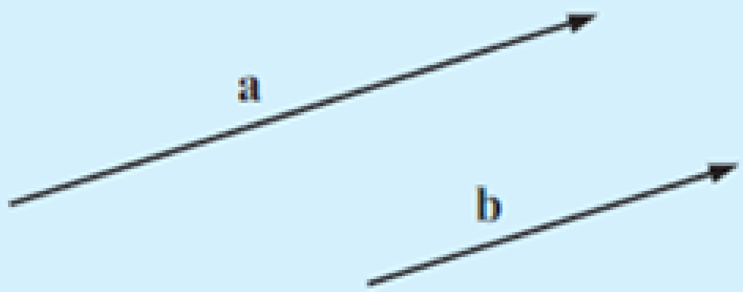
12I – Parallelism

If two vectors are parallel, then one is a scalar multiple of the other.

Given any non-zero vector $\mathbf{v}$ and non-zero scalar k, the vector $k\mathbf{v}$ is parallel to $\mathbf{v}$.



- If $\mathbf{a}$ is parallel to $\mathbf{b}$, then there exists a scalar k such that $\mathbf{a} = k\mathbf{b}$.
- If $\mathbf{a} = k\mathbf{b}$ for some scalar k, then
  - $\mathbf{a}$ is parallel to $\mathbf{b}$, and
  - $|\mathbf{a}| = |k|\,|\mathbf{b}|$.

Example:

1. If $\mathrm{a} = \begin{pmatrix} 15 \\ 10 \end{pmatrix}$ and $\mathrm{b} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ then **a** is parallel to **b**.

Find: $|\mathbf{a}|$ and $|\mathbf{b}|$

$$|\vec{a}| = \sqrt{(15)^2 + (10)^2}$$
$$= \sqrt{225 + 100}$$
$$= \sqrt{325}$$
$$= 5\sqrt{13}$$

$$|\vec{b}| = \sqrt{(3)^2 + (2)^2}$$
$$= \sqrt{9+4}$$
$$= \sqrt{13}$$

$$|\vec{a}| = 5|\vec{b}|$$

2. Find r and s given that $\mathbf{a} = \begin{pmatrix} 2 \\ -1 \\ r \end{pmatrix}$ is parallel to $\mathbf{b} = \begin{pmatrix} 8 \\ s \\ -6 \end{pmatrix}$.

$k\vec{a} = \vec{b}$

$k\begin{pmatrix} 2 \\ -1 \\ r \end{pmatrix} = \begin{pmatrix} 8 \\ s \\ -6 \end{pmatrix}$

$2k = 8$

$k = 4$

$(-1)(k) = s$

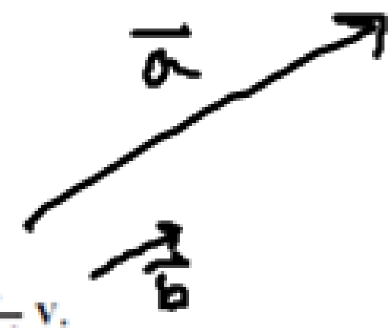
$-4 = s$

$kr = -6$

$4r = -6$

$r = \frac{-3}{2}$

Unit Vectors:



Given a non-zero vector $\mathbf{v}$, its magnitude $|\mathbf{v}|$ is a scalar quantity.

If we multiply $\mathbf{v}$ by the scalar $\dfrac{1}{|\mathbf{v}|}$, we obtain the parallel vector $\dfrac{1}{|\mathbf{v}|}\mathbf{v}$.

The length of this vector is $\left|\dfrac{1}{|\mathbf{v}|}\right||\mathbf{v}| = \dfrac{|\mathbf{v}|}{|\mathbf{v}|} = 1$, so $\dfrac{1}{|\mathbf{v}|}\mathbf{v}$ is a unit vector in the direction of $\mathbf{v}$.

- A unit vector in the direction of $\mathbf{v}$ is $\dfrac{1}{|\mathbf{v}|}\mathbf{v}$.
- A vector $\mathbf{b}$ of length k in the same direction as $\mathbf{a}$ is $\mathbf{b} = \dfrac{k}{|\mathbf{a}|}\mathbf{a}$.
- A vector $\mathbf{b}$ of length k which is *parallel to* $\mathbf{a}$ could be $\mathbf{b} = \pm\dfrac{k}{|\mathbf{a}|}\mathbf{a}$.

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$\vec{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $\vec{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

6 Find the unit vector in the direction of:

~~a~~ i + 2j   b 2i − 3k   c 2i − 2j + k

$\begin{pmatrix} 2 \\ 0 \\ -3 \end{pmatrix}$

length : $|2\vec{i} - 3\vec{k}| = \sqrt{(2)^2 + (0)^2 + (-3)^2}$

$= \sqrt{4+9}$

$= \sqrt{13}$

unit vector $= \frac{1}{\sqrt{13}}(2\vec{i} - 3\vec{k})$

10 Find a vector **b** in:

a the same direction as $\vec{v} = \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}$ and with length 6 units

① find length of $\vec{v}$

$$|\vec{v}| = \sqrt{(-1)^2 + (4)^2 + (1)^2}$$
$$= \sqrt{1+16+1}$$
$$= \sqrt{18}$$
$$= 3\sqrt{2}$$

② unit vector in the same direction $= \frac{1}{3\sqrt{2}} \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}$

$$\frac{2}{\sqrt{2}} \left(\frac{\sqrt{2}}{\sqrt{2}}\right) = \frac{2\sqrt{2}}{2}$$

③ vector of length 6:

$$\frac{6}{3\sqrt{2}} \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix} = \frac{2}{\sqrt{2}} \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}$$
$$= \sqrt{2} \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} -\sqrt{2} \\ 4\sqrt{2} \\ \sqrt{2} \end{pmatrix}$$

12J – The Scalar Product of Two Vectors

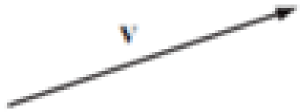
- The **scalar product** of 2 vectors, which results in a **scalar** answer and has the notation $\mathbf{v} \bullet \mathbf{w}$ (read "v dot w").
- The **vector product** of 2 vectors, which results in a **vector** answer and has the notation $\mathbf{v} \times \mathbf{w}$ (read "v cross w").

The **scalar product** of two vectors is also known as the **dot product** or **inner product**.

If $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$, the **scalar product** of $\mathbf{v}$ and $\mathbf{w}$ is defined as

$\mathbf{v} \bullet \mathbf{w} = v_1 w_1 + v_2 w_2 + v_3 w_3.$

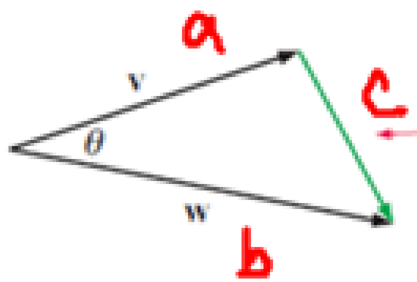
Angle Between Vectors:

Consider the vectors 

$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ and 

$\mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$.

We translate one of the vectors so that they both originate from the same point.



This vector is $-\mathbf{v} + \mathbf{w} = \mathbf{w} - \mathbf{v}$ and has length $|\mathbf{w} - \mathbf{v}|$.

Using the cosine rule, $|\mathbf{w} - \mathbf{v}|^2 = |\mathbf{v}|^2 + |\mathbf{w}|^2 - 2|\mathbf{v}||\mathbf{w}|\cos\theta$

But $\mathbf{w} - \mathbf{v} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} - \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} w_1 - v_1 \\ w_2 - v_2 \\ w_3 - v_3 \end{pmatrix}$

$$\therefore \ (w_1 - v_1)^2 + (w_2 - v_2)^2 + (w_3 - v_3)^2 = v_1^2 + v_2^2 + v_3^2 + w_1^2 + w_2^2 + w_3^2 - 2|\mathbf{v}||\mathbf{w}|\cos\theta$$

$$\therefore \ \underline{v_1 w_1 + v_2 w_2 + v_3 w_3} = |\mathbf{v}||\mathbf{w}|\cos\theta$$

$$\therefore \ \mathbf{v} \bullet \mathbf{w} = |\mathbf{v}||\mathbf{w}|\cos\theta$$

The angle θ between two vectors $\mathbf{v}$ and $\mathbf{w}$ can be found using

$$\cos\theta = \frac{\mathbf{v} \bullet \mathbf{w}}{|\mathbf{v}||\mathbf{w}|}$$

ALGEBRAIC PROPERTIES OF THE SCALAR PRODUCT

The scalar product has the following algebraic properties for both 2-D and 3-D vectors:

- $\mathbf{v} \bullet \mathbf{w} = \mathbf{w} \bullet \mathbf{v}$
- $\mathbf{v} \bullet \mathbf{v} = |\mathbf{v}|^2$
- $\mathbf{v} \bullet (\mathbf{w} + \mathbf{x}) = \mathbf{v} \bullet \mathbf{w} + \mathbf{v} \bullet \mathbf{x}$
- $(\mathbf{v} + \mathbf{w}) \bullet (\mathbf{x} + \mathbf{y}) = \mathbf{v} \bullet \mathbf{x} + \mathbf{v} \bullet \mathbf{y} + \mathbf{w} \bullet \mathbf{x} + \mathbf{w} \bullet \mathbf{y}$

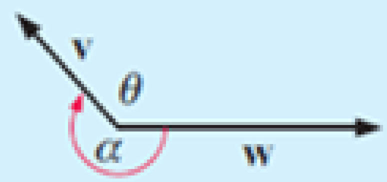
GEOMETRIC PROPERTIES OF THE SCALAR PRODUCT

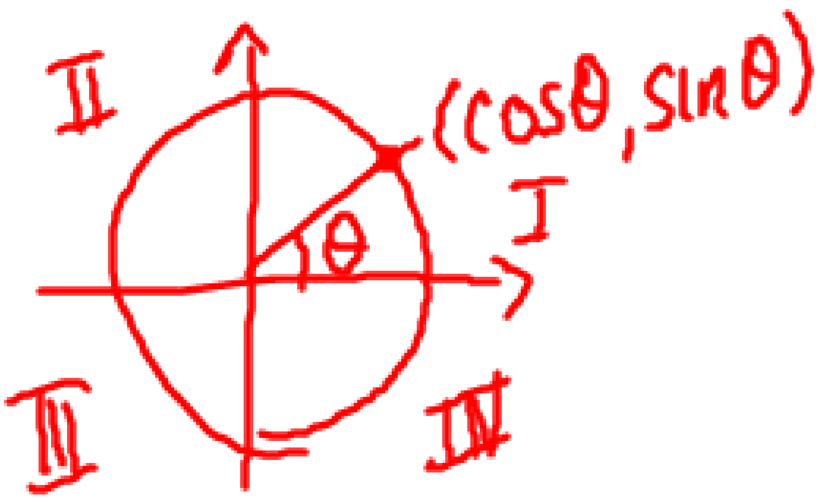
- For non-zero vectors $\mathbf{v}$ and $\mathbf{w}$:
  $\mathbf{v} \bullet \mathbf{w} = 0 \quad \Leftrightarrow \quad$ $\mathbf{v}$ and $\mathbf{w}$ are perpendicular or orthogonal.
- $|\mathbf{v} \bullet \mathbf{w}| = |\mathbf{v}||\mathbf{w}| \quad \Leftrightarrow \quad$ $\mathbf{v}$ and $\mathbf{w}$ are non-zero parallel vectors.
- If θ is the angle between vectors $\mathbf{v}$ and $\mathbf{w}$ then: $\mathbf{v} \bullet \mathbf{w} = |\mathbf{v}||\mathbf{w}|\cos\theta$

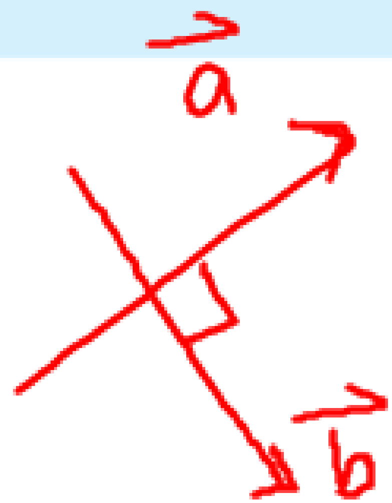
  If θ is acute, $\cos\theta > 0$ and so $\mathbf{v} \bullet \mathbf{w} > 0$

  If θ is obtuse, $\cos\theta < 0$ and so $\mathbf{v} \bullet \mathbf{w} < 0$.

  The angle between two vectors is always taken as the angle θ such that $0° \leqslant \theta \leqslant 180°$, rather than reflex angle α.







The first two of these results can be demonstrated as follows:

If $\mathbf{v}$ is perpendicular to $\mathbf{w}$ then $\theta = 90°$.

$$\begin{aligned} \therefore \quad \mathbf{v} \bullet \mathbf{w} &= |\mathbf{v}||\mathbf{w}|\cos\theta \\ &= |\mathbf{v}||\mathbf{w}|\cos 90° \\ &= 0 \end{aligned}$$

If $\mathbf{v}$ is parallel to $\mathbf{w}$ then $\theta = 0°$ or $180°$.

$$\begin{aligned} \therefore \quad \mathbf{v} \bullet \mathbf{w} &= |\mathbf{v}||\mathbf{w}|\cos\theta \\ &= |\mathbf{v}||\mathbf{w}|\cos 0° \quad \text{or} \quad |\mathbf{v}||\mathbf{w}|\cos 180° \\ &= \pm|\mathbf{v}||\mathbf{w}| \end{aligned}$$

$$\therefore \quad |\mathbf{v} \bullet \mathbf{w}| = |\mathbf{v}||\mathbf{w}|$$

To formally prove these results we must also show that their converses are true.

Examples:

1. Find the scalar product of $\vec{p} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and $\vec{q} = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$.

$$\vec{p} \cdot \vec{q} = (1)(-2) + (3)(5)$$
$$= -2 + 15$$
$$= 13$$

2. Find m such that $\vec{p} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$ and $\vec{q} = \begin{pmatrix} 1 \\ m \end{pmatrix}$ are perpendicular.

$$\vec{p} \cdot \vec{q} = 0$$

$$(-2)(1) + (3)(m) = 0$$

$$3m = 2$$

$$m = \frac{2}{3}$$

4. Find $\mathbf{a} \cdot \mathbf{b}$ for:

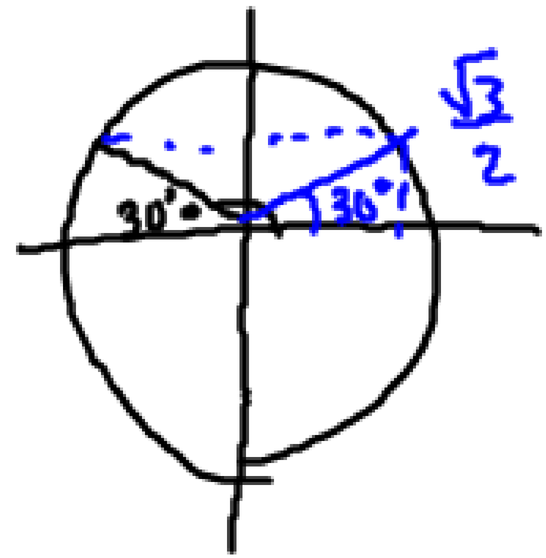
(a) $|\mathbf{a}| = 4, |\mathbf{b}| = 5$, and $\theta = 150°$

$\vec{a} \cdot \vec{b} = |a||b| \cos\theta$

$\vec{a} \cdot \vec{b} = (4)(5)\cos 150°$ ← unit circle value

$= 20\left(-\frac{\sqrt{3}}{2}\right)$

$\vec{a} \cdot \vec{b} = -10\sqrt{3}$



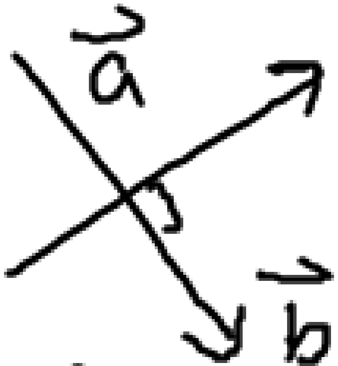
(b) $|\mathbf{a}| = 3, |\mathbf{b}| = 2$, and $\theta = 75°$ ← not a unit circle

$\vec{a} \cdot \vec{b} = (3)(2)\cos 75°$

$= 6(0.2588)$

$= 1.5529$

3. Find the form of all vectors which are perpendicular to $\begin{pmatrix} 3 \\ 7 \end{pmatrix} = \vec{a}$.



or

$\vec{a} \bullet \vec{b} = 0$

$\begin{pmatrix} 3 \\ 7 \end{pmatrix} \bullet \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = 0$

$3b_1 + 7b_2 = 0$

$b_1 = 7 \therefore b_2 = -3$

or

$b_1 = -7 \therefore b_2 = 3$

$\vec{b} = \begin{pmatrix} 7 \\ -3 \end{pmatrix}$

$\vec{a} \bullet \vec{b} = (3)(7) + (7)(-3)$

$= 21 - 21$

$= 0$

$k\begin{pmatrix} 7 \\ -3 \end{pmatrix}$, $k \neq 0$ is perp. to $\begin{pmatrix} 3 \\ 7 \end{pmatrix}$

$m\begin{pmatrix} -7 \\ 3 \end{pmatrix}$, $m \neq 0$ is per. to $\begin{pmatrix} 3 \\ 7 \end{pmatrix}$

5. Find the angle between $\vec{a} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 2 \\ 7 \end{pmatrix}$.

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$$

$$\cos\theta = \frac{(4)(2) + (-3)(7)}{\left(\sqrt{(4)^2 + (-3)^2}\right)\left(\sqrt{(2)^2 + (7)^2}\right)}$$

$$= \frac{8 - 21}{(\sqrt{25})(\sqrt{53})}$$

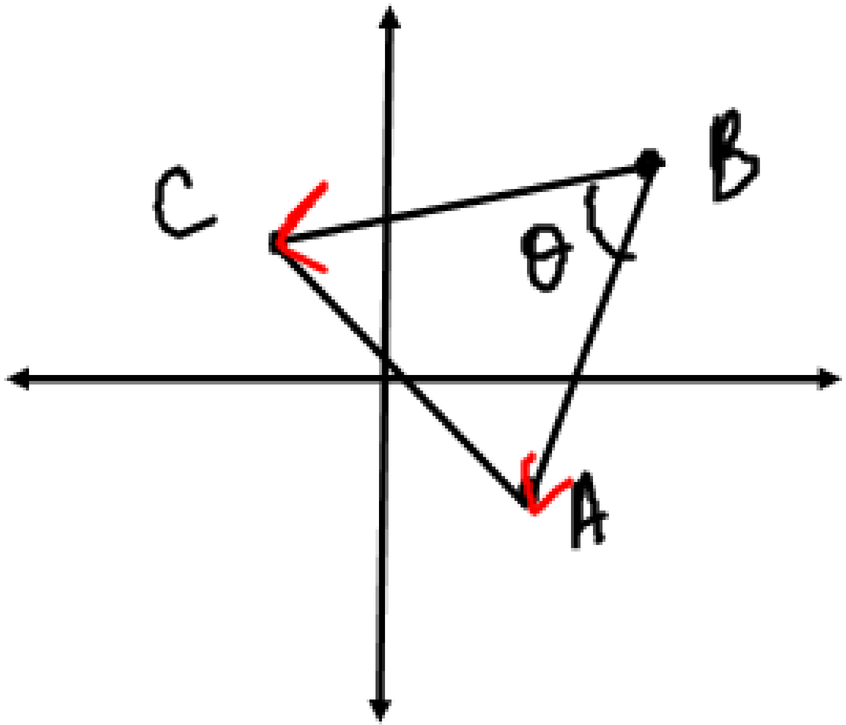
$$= \frac{-13}{5\sqrt{53}}$$

$$\cos\theta = \frac{-13}{5\sqrt{53}}$$

$$\cos\theta = -0.357137$$

$$\theta = 110.924^\circ$$

6. Find the measure of angle ABC for A(2, -1), B(3, 4) and C(-1, 3).



$$\overrightarrow{BA} = \begin{pmatrix} 2-3 \\ -1-4 \end{pmatrix} = \begin{pmatrix} -1 \\ -5 \end{pmatrix}$$

$$\overrightarrow{BC} = \begin{pmatrix} -1-3 \\ 3-4 \end{pmatrix} = \begin{pmatrix} -4 \\ -1 \end{pmatrix}$$

$$\cos\theta = \frac{\overrightarrow{BA} \cdot \overrightarrow{BC}}{|\overrightarrow{BA}||\overrightarrow{BC}|}$$

$$= \frac{(-1)(-4) + (-5)(-1)}{\left(\sqrt{(1)^2 + (-5)^2}\right)\left(\sqrt{(-4)^2 + (-1)^2}\right)}$$

$$= \frac{4+5}{\sqrt{26}\sqrt{17}}$$

$$= \frac{9}{\sqrt{26} \cdot \sqrt{17}}$$

$$\theta = \cos^{-1}(0.42808)$$

$$\theta = 64.65^\circ$$