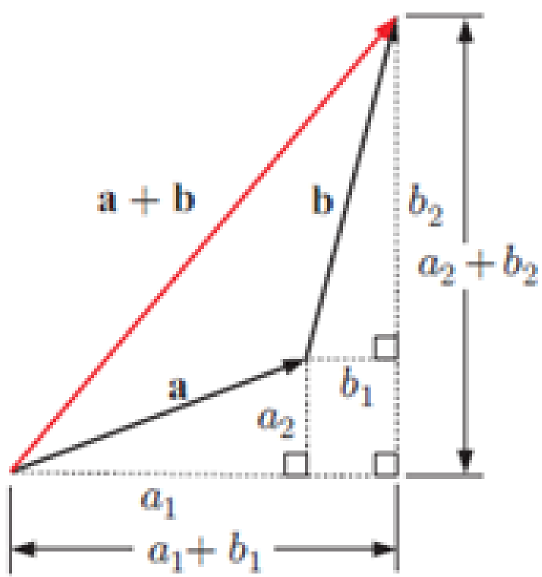
12E – Operations with Plane Vectors

Vector Addition:

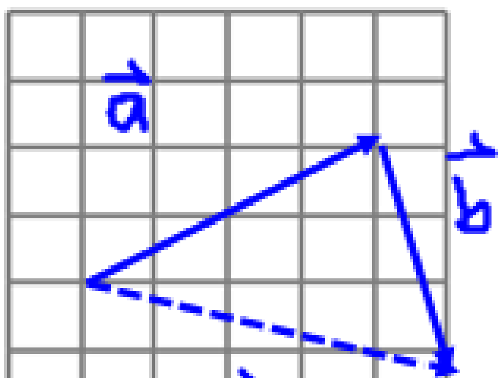
If $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$, then $\mathbf{a} + \mathbf{b} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \end{pmatrix}$



Example:

If $\vec{a} = \begin{pmatrix}4\\2\end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix}1\\-3\end{pmatrix}$, find $\mathbf{a} + \mathbf{b}$.

$$\vec{a} + \vec{b} = \begin{pmatrix}4\\2\end{pmatrix} + \begin{pmatrix}1\\-3\end{pmatrix} = \begin{pmatrix}5\\-1\end{pmatrix}$$



$\vec{a}+\vec{b} = \begin{pmatrix}5\\-1\end{pmatrix}$

Negative Vectors:

$\mathbf{a} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$ $\qquad$ $-\mathbf{a} = ?$

$= \begin{pmatrix} -2 \\ 3 \end{pmatrix}$

$-\begin{pmatrix} 2 \\ -3 \end{pmatrix}$

If $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ then $-\mathbf{a} = \begin{pmatrix} -a_1 \\ -a_2 \end{pmatrix}$

Vector Subtraction – add its negative

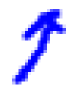
$$\mathbf{a} - \mathbf{b} = \mathbf{a} + (\mathbf{-b})$$

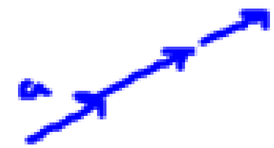
If $\mathbf{a} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$, find $\mathbf{a} - \mathbf{b}$.

$$\begin{pmatrix} 4 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ -3 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$$

Scalar Multiplication

$\mathbf{a} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ $\qquad 2\mathbf{a} = \mathbf{a} + \mathbf{a} \qquad 3\mathbf{a} = \mathbf{a} + \mathbf{a} + \mathbf{a}$





If k is a scalar, then $k\mathbf{a} = \begin{pmatrix} ka_1 \\ ka_2 \end{pmatrix}$

$\vec{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \vec{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Example:

For $\mathbf{p} = \mathbf{i} - 2\mathbf{j}$ and $\mathbf{q} = 3\mathbf{i} + 4\mathbf{j}$, find:

(a) $-2\mathbf{p}$

$-2\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$

(b) $\frac{1}{2}\mathbf{p} - 2\mathbf{q}$

$\frac{1}{2}(\vec{i} - 2\vec{j}) - 2(3\vec{i} + 4\vec{j})$

$\frac{1}{2}\vec{i} - \vec{j} - 6\vec{i} - 8\vec{j}$

$-5.5\vec{i} - 9\vec{j}$

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7 If $\mathbf{p} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$ find:

a $|\mathbf{p}|$ ← magnitude length **b** $|2\mathbf{p}|$

$|\vec{p}|$

$= \sqrt{(1)^2+(3)^2}$

$= \sqrt{1+9}$

$|\vec{p}| = \sqrt{10}$

$2\vec{p} = \begin{pmatrix} 2 \\ 6 \end{pmatrix}$

$= \sqrt{(2)^2+(6)^2}$

$= \sqrt{4+36}$

$= \sqrt{40} = \sqrt{4 \cdot 10} = \sqrt{4}\sqrt{10}$

$= 2\sqrt{10}$

$|2\vec{p}| = 2|\vec{p}|$

12F – The Vector Between Two Points

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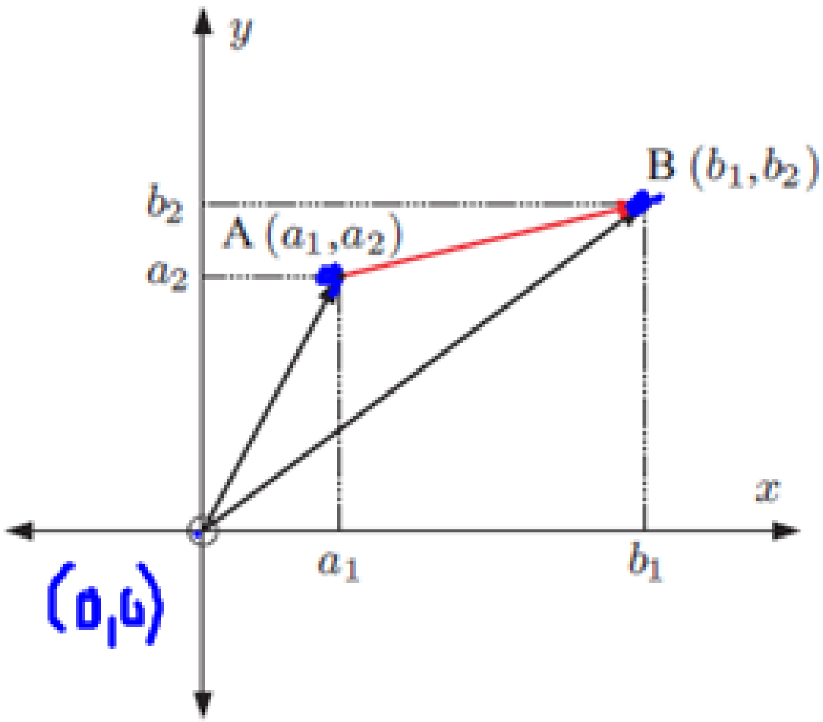

The position vector of B relative to A is $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \end{pmatrix}$.

Example:

Given points A(2, 3) and B(-4, 1), give the position vector of:

(a) B from O

$$\begin{pmatrix} -4 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -4 \\ 1 \end{pmatrix}$$

(b) B from A $\overrightarrow{AB}$

$$\begin{pmatrix} -4 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -6 \\ -2 \end{pmatrix}$$

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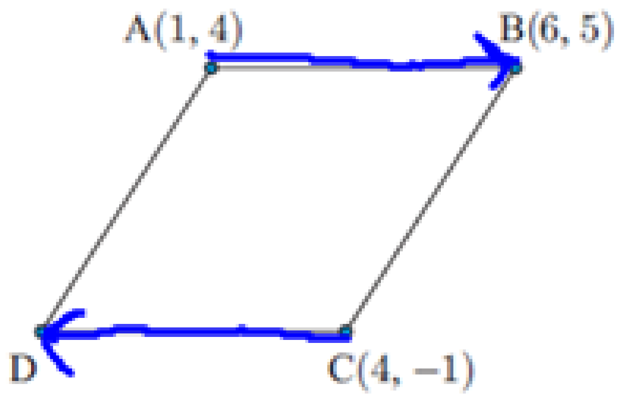


ABCD is a parallelogram.

a Find $\overrightarrow{AB}$.

b Find $\overrightarrow{CD}$.

c Hence find the coordinates of D.

A) $\overrightarrow{AB}$

$\begin{pmatrix} 6 \\ 5 \end{pmatrix} - \begin{pmatrix} 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$

B) $\overrightarrow{CD} = -\overrightarrow{AB}$

$= \begin{pmatrix} -5 \\ -1 \end{pmatrix}$

c) $\overrightarrow{CD} = \begin{pmatrix} d_1 - 4 \\ d_2 - (-1) \end{pmatrix} = \begin{pmatrix} -5 \\ -1 \end{pmatrix}$

$d_1 - 4 = -5 \quad \therefore d_1 = -1$

$d_2 + 1 = -1 \quad \therefore d_2 = -2$

$D(-1, -2)$

5 A(−1, 3) and B(3, k) are two points which are 5 units apart.

a Find $\overrightarrow{AB}$ and $|\overrightarrow{AB}|$.

b Hence, find the two possible values of k.

c Show, by illustration, why k should have two possible values.

A) $\overrightarrow{AB} = \begin{pmatrix} 3-(-1) \\ K-3 \end{pmatrix}$

$= \begin{pmatrix} 4 \\ K-3 \end{pmatrix}$

$|\overrightarrow{AB}| = 5$

B) $|\overrightarrow{AB}| = \sqrt{(4)^2 + (K-3)^2}$

$5 = \sqrt{16+(K-3)^2}$

$25 = 16+(K-3)^2$

$9 = (K-3)^2$

$\sqrt{9} = \sqrt{(K-3)^2}$

$\pm 3 = K-3$

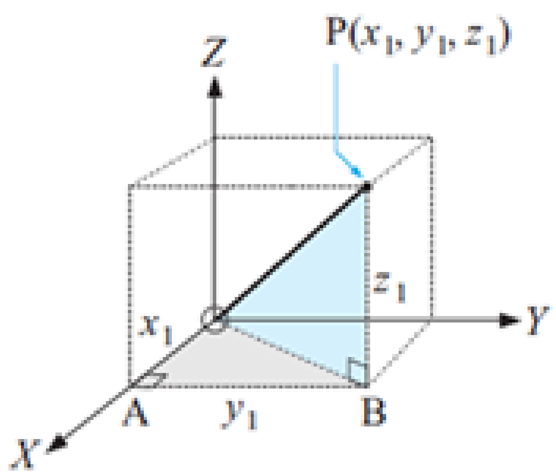
$K=0 \quad K=6$

12G – Vectors in Space

Any point P in space can be specified by an ordered triple of numbers (x, y, z) where x, y, z are the steps in the X, Y, and Z directions from the origin, O, to P.

$$\vec{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \vec{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \vec{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

The position vector of P is: $$\vec{OP} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$



The **magnitude** of the vector

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \text{ is } |\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$$

THE VECTOR BETWEEN TWO POINTS

If $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ are two points in space then:

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{pmatrix} \begin{array}{l} \leftarrow x\text{-step} \\ \leftarrow y\text{-step} \\ \leftarrow z\text{-step} \end{array}$$

$\overrightarrow{AB}$ is called the 'vector AB' or the '**position vector of B relative to A**'.

The magnitude of $\overrightarrow{AB}$ is $AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$ which is the distance between the points A and B.

Example:

If A is (3, -1, 2) and B is (1, 0, -2) find:

(a) $\overrightarrow{\mathbf{OA}}$

$$\begin{pmatrix} 3 - 0 \\ -1 - 0 \\ 2 - 0 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$$

(b) $\overrightarrow{\mathbf{AB}}$

B − A

$$\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} - \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ -4 \end{pmatrix}$$

(c) $|\overrightarrow{\mathbf{AB}}|$

$$= \sqrt{(-2)^2 + (1)^2 + (-4)^2}$$

$$= \sqrt{4 + 1 + 16}$$

$$= \sqrt{21}$$

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14 Find k given the unit vector:

a $\begin{pmatrix} -\frac{1}{2} \\ k \\ \frac{1}{4} \end{pmatrix} = \vec{v}$

$|\vec{v}| = \sqrt{\left(-\frac{1}{2}\right)^2 + (k)^2 + \left(\frac{1}{4}\right)^2}$

$1 = \sqrt{\frac{1}{4} + k^2 + \frac{1}{16}}$

$(1)^2 = \left(\sqrt{\frac{5}{16} + k^2}\right)^2$

$1 = \frac{5}{16} + k^2$

$k^2 = \frac{16}{16} - \frac{5}{16}$

$k^2 = \frac{11}{16}$

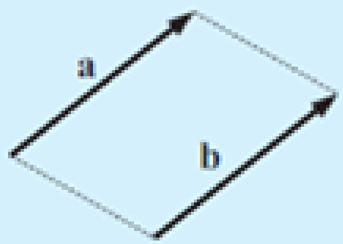
$k = \pm\sqrt{\frac{11}{16}}$

$= \pm\frac{\sqrt{11}}{4}$

VECTOR EQUALITY

Two vectors are **equal** if they have the same magnitude and direction.

If $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$, then $\mathbf{a} = \mathbf{b} \;\Leftrightarrow\; a_1 = b_1,\; a_2 = b_2,\; a_3 = b_3.$



If **a** and **b** do not coincide, then they are opposite sides of a parallelogram, and lie in the same plane.

Find a, b, and c if: $\begin{pmatrix} a-3 \\ 2b-1 \\ c+2 \end{pmatrix} = \begin{pmatrix} 4 \\ -b \\ 2c \end{pmatrix}$

$a-3=4 \therefore a=7$

$2b-1=-b \therefore 3b=1, b=\frac{1}{3}$

$c+2=2c \therefore c=2.$

15 A(−1, 3, 4), B(2, 5, −1), C(−1, 2, −2), and D(r, s, t) are four points in space.

Find r, s, and t if: **a** $\overrightarrow{AC} = \overrightarrow{BD}$ **b** ~~$\overrightarrow{AB} = \overrightarrow{DC}$~~

$$\overrightarrow{AC} = \begin{pmatrix} -1-(-1) \\ 2-3 \\ -2-4 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ -6 \end{pmatrix}$$

$$\overrightarrow{BD} = \begin{pmatrix} r-2 \\ s-5 \\ t-(-1) \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ -1 \\ -6 \end{pmatrix} = \begin{pmatrix} r-2 \\ s-5 \\ t+1 \end{pmatrix}$$

$r-2=0 \quad \therefore r=2$

$s-5=-1 \quad \therefore s=4$

$t+1=-6 \quad \therefore t=-7$

12H – Operations with Vectors in Space

- Similar to operations with 2-D vectors

If $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ then $\mathbf{a} + \mathbf{b} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{pmatrix}$, $\mathbf{a} - \mathbf{b} = \begin{pmatrix} a_1 - b_1 \\ a_2 - b_2 \\ a_3 - b_3 \end{pmatrix}$,

and $k\mathbf{a} = \begin{pmatrix} ka_1 \\ ka_2 \\ ka_3 \end{pmatrix}$ for any scalar k.

- $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$ {commutative property}
- $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$ {associative property}
- $\mathbf{a} + \mathbf{0} = \mathbf{0} + \mathbf{a} = \mathbf{a}$ {additive identity}
- $\mathbf{a} + (-\mathbf{a}) = (-\mathbf{a}) + \mathbf{a} = \mathbf{0}$ {additive inverse}
- $\underbrace{|k\mathbf{a}|}_{\text{length of } k\mathbf{a}} = \underbrace{|k|}_{\text{modulus of } k}\ \underbrace{|\mathbf{a}|}_{\text{length of } \mathbf{a}}$ where $k\mathbf{a}$ is parallel to $\mathbf{a}$
- $k(\mathbf{a} + \mathbf{b}) = k\mathbf{a} + k\mathbf{b}$ {distributive property}

- if $\mathbf{x} + \mathbf{a} = \mathbf{b}$ then $\mathbf{x} = \mathbf{b} - \mathbf{a}$
- if $k\mathbf{x} = \mathbf{a}$ then $\mathbf{x} = \frac{1}{k}\mathbf{a}$ $(k \neq 0)$

Examples:

1. For $\mathbf{a} = \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix}$ find:

(a) $\mathbf{a} - \mathbf{b}$

$$\begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix} - \begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -5 \\ 6 \end{pmatrix}$$

(b) $\mathbf{b} + 2\mathbf{a}$

$$\begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix} + \begin{pmatrix} 2 \\ -4 \\ 10 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 9 \end{pmatrix}$$

(c) $|\mathbf{a}| = \sqrt{(1)^2 + (-2)^2 + (5)^2}$

$$= \sqrt{1 + 4 + 25}$$

$$= \sqrt{30}$$

(d) $|\mathbf{a} + \mathbf{b}|$

$$\vec{a} + \vec{b} = \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix} + \begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}$$

$$= \sqrt{(1)^2 + (1)^2 + (4)^2}$$

$$= \sqrt{1 + 1 + 16}$$

$$= \sqrt{18}$$

$$= 3\sqrt{2}$$

Examples:

1. For $\mathbf{a} = \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix}$ find:

(e) $\mathbf{x}$ if $\quad \frac{1}{2}\mathbf{x} = \mathbf{b}$

$$\vec{x} = 2\vec{b}$$

$$\vec{x} = 2\begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 6 \\ -2 \end{pmatrix}$$

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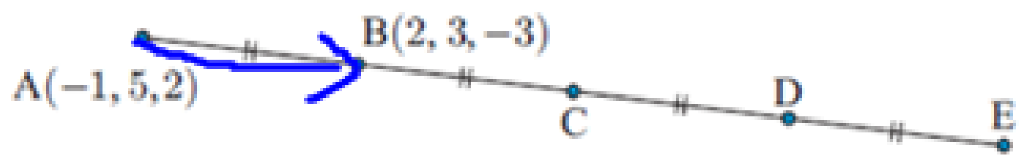
8 Find the coordinates of C, D, and E.



$$\overrightarrow{AB} = \begin{pmatrix} 2-(-1) \\ 3-5 \\ -3-2 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ -5 \end{pmatrix}$$

$$C = (c_1, c_2, c_3)$$
$$= (2+3, 3+-2, -3+-5)$$
$$= (5, 1, -8)$$

$$D = (d_1, d_2, d_3)$$
$$= (5+3, 1+-2, -5+-8)$$
$$= (8, -1, -13)$$

$$E = (e_1, e_2, e_3)$$
$$= (8+3, -1+-2, -13+-5)$$
$$= (11, -3, -18)$$