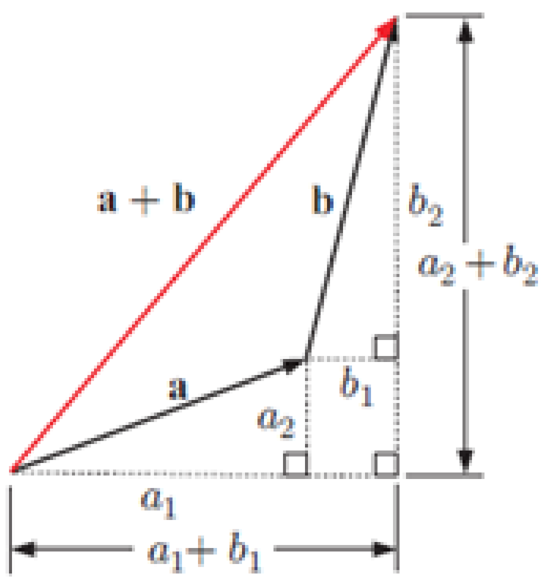
12E – Operations with Plane Vectors

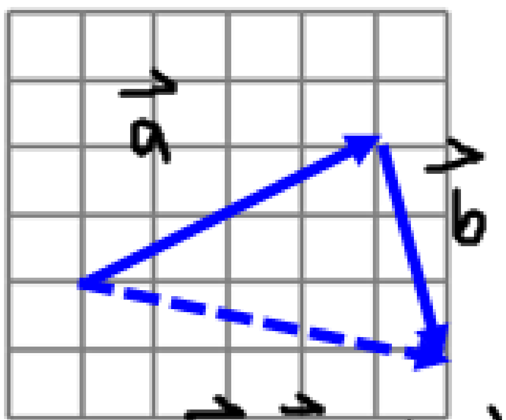
Vector Addition:

If $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$, then $\mathbf{a} + \mathbf{b} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \end{pmatrix}$



Example:

If $\vec{a} = \begin{pmatrix}4\\2\end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix}1\\-3\end{pmatrix}$, find $\mathbf{a}+\mathbf{b}$.



$\vec{a}+\vec{b} = \begin{pmatrix}5\\-1\end{pmatrix}$

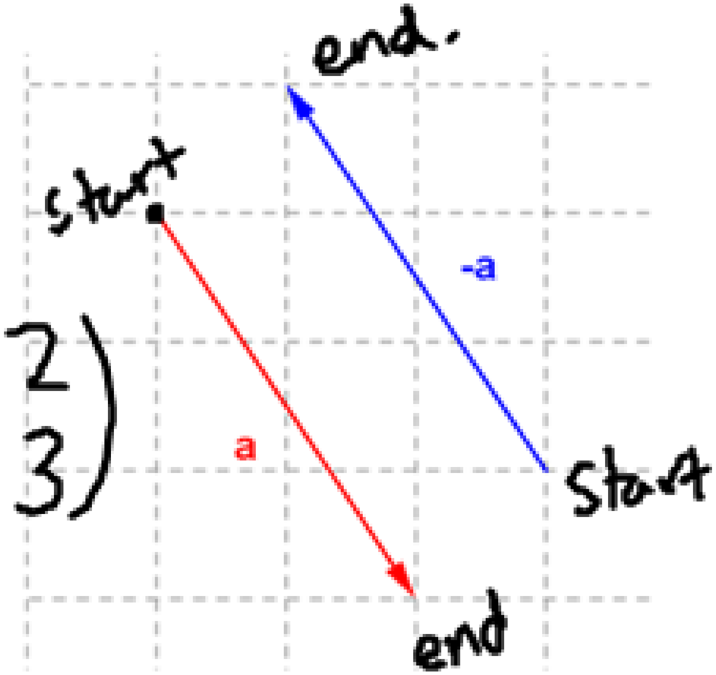
$\vec{a}+\vec{b}$

$\begin{pmatrix}4\\2\end{pmatrix}+\begin{pmatrix}1\\-3\end{pmatrix}=\begin{pmatrix}5\\-1\end{pmatrix}$

Negative Vectors:

$$\mathbf{a} = \begin{pmatrix} 2 \\ -3 \end{pmatrix} \qquad \mathbf{-a} = ?$$

$$-\begin{pmatrix} 2 \\ -3 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$$



If $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ then $-\mathbf{a} = \begin{pmatrix} -a_1 \\ -a_2 \end{pmatrix}$

Vector Subtraction – add its negative

$$\mathbf{a} - \mathbf{b} = \mathbf{a} + (\mathbf{-b})$$

If $\mathbf{a} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$, find $\mathbf{a} - \mathbf{b}$.

$$\begin{pmatrix} 4 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ -3 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$$

Scalar Multiplication

$\mathbf{a} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ $\qquad 2\mathbf{a} = \mathbf{a} + \mathbf{a} \qquad 3\mathbf{a} = \mathbf{a} + \mathbf{a} + \mathbf{a}$

If k is a scalar, then $k\mathbf{a} = \begin{pmatrix} ka_1 \\ ka_2 \end{pmatrix}$

$\vec{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \vec{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Example:

For $\mathbf{p} = \mathbf{i} - 2\mathbf{j}$ and $\mathbf{q} = 3\mathbf{i} + 4\mathbf{j}$, find:

(a) $-2\mathbf{p}$

$-2\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$

(b) $\frac{1}{2}\mathbf{p} - 2\mathbf{q}$

$= \frac{1}{2}(\vec{i} - 2\vec{j}) - 2(3\vec{i} + 4\vec{j})$

$= \frac{1}{2}\vec{i} - 6\vec{i} - 1\vec{j} - 8\vec{j}$

$= -5{,}5\vec{i} - 9\vec{j}$

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7 If $\mathbf{p} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$ find:

a $|\mathbf{p}|$ **b** $|2\mathbf{p}|$

$$2\vec{p} = \begin{pmatrix} 2 \\ 6 \end{pmatrix}$$

$$|\vec{p}| = \sqrt{(1)^2 + (3)^2}$$
$$= \sqrt{10}$$

$$|2\vec{p}| = \sqrt{(2)^2 + (6)^2}$$
$$= \sqrt{4+36}$$
$$= \sqrt{40}$$
$$= 2\sqrt{10}$$

12F – The Vector Between Two Points

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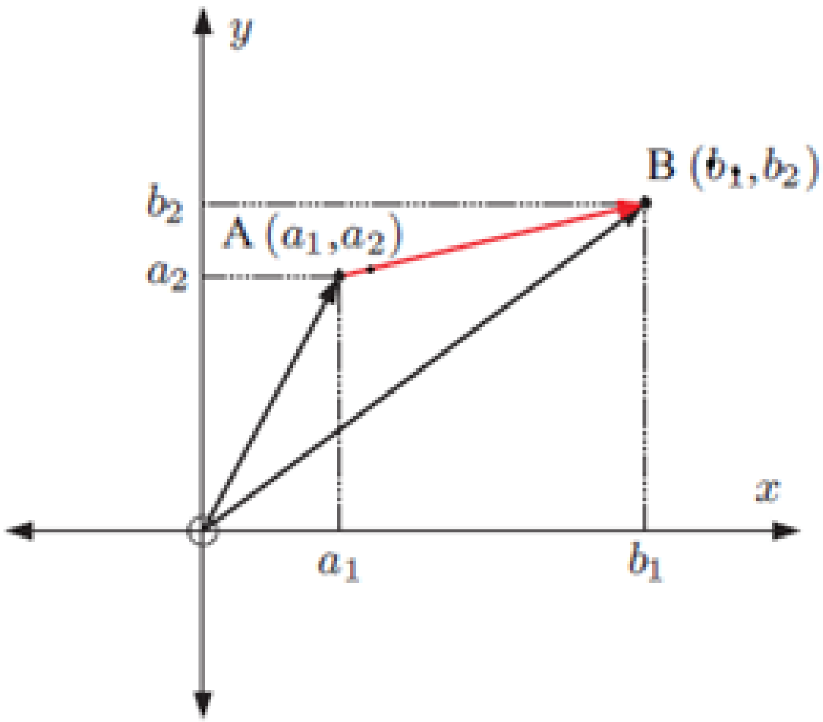

The position vector of **B** relative to **A** is $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \end{pmatrix}$.

Example:

Given points A(2, 3) and B(-4, 1), give the position vector of:

(a) B from O

$$\begin{pmatrix} -4 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -4 \\ 1 \end{pmatrix}$$

(b) B from A

$$\begin{pmatrix} -4 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -6 \\ -2 \end{pmatrix}$$

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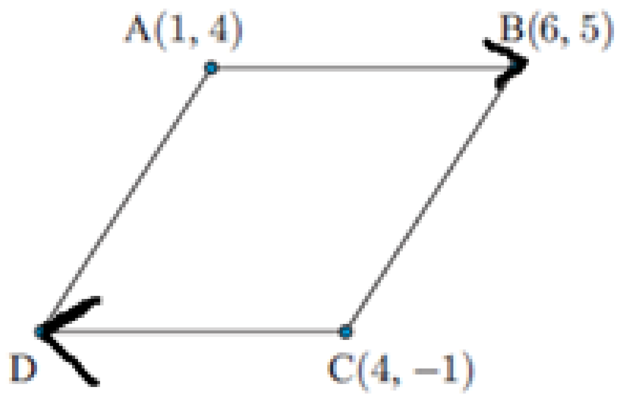


ABCD is a parallelogram.

a Find $\overrightarrow{AB}$.

b Find $\overrightarrow{CD}$.

c Hence find the coordinates of D.

A) $\overrightarrow{AB} = \begin{pmatrix}6\\5\end{pmatrix} - \begin{pmatrix}1\\4\end{pmatrix} = \begin{pmatrix}5\\1\end{pmatrix}$

B) $\overrightarrow{CD} = -\overrightarrow{AB}$

$= -\begin{pmatrix}5\\1\end{pmatrix} = \begin{pmatrix}-5\\-1\end{pmatrix}$

c) $\overrightarrow{CD} = \begin{pmatrix}-5\\-1\end{pmatrix} = \begin{pmatrix}d_1\\d_2\end{pmatrix} - \begin{pmatrix}4\\-1\end{pmatrix}$

$-5 = d_1 - 4 \therefore d_1 = -1$

$-1 = d_2 - (-1) \therefore d_2 = -2$

$D(-1, -2)$

5 A(−1, 3) and B(3, k) are two points which are 5 units apart.

a Find $\overrightarrow{AB}$ and $|\overrightarrow{AB}|$.

b Hence, find the two possible values of k.

c Show, by illustration, why k should have two possible values.

A) $\overrightarrow{AB} = \begin{pmatrix} 3-(-1) \\ k-3 \end{pmatrix} = \begin{pmatrix} 4 \\ k-3 \end{pmatrix}$ $|\overrightarrow{AB}| = 5$

$|\overrightarrow{AB}| = \sqrt{(4)^2+(k-3)^2}$

$5 = \sqrt{16+(k-3)^2}$

$25 = 16+(k-3)^2$

$9 = (k-3)^2$

$\sqrt{9} = \sqrt{(k-3)^2}$

$\pm 3 = k-3$

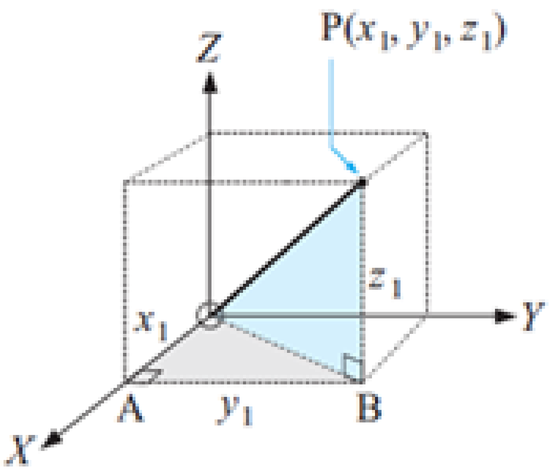
$k = 0$

$k = 6$

12G – Vectors in Space

Any point P in space can be specified by an ordered triple of numbers (x, y, z) where x, y, z are the steps in the X, Y, and Z directions from the origin, O, to P.

The position vector of P is: $$\vec{OP} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$



The **magnitude** of the vector

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \text{ is } |\mathbf{v}| = \sqrt{{v_1}^2 + {v_2}^2 + {v_3}^2}$$

THE VECTOR BETWEEN TWO POINTS

If $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ are two points in space then:

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{pmatrix} \begin{matrix} \leftarrow x\text{-step} \\ \leftarrow y\text{-step} \\ \leftarrow z\text{-step} \end{matrix}$$

$\overrightarrow{AB}$ is called the 'vector AB' or the '**position vector of B relative to A**'.

The magnitude of $\overrightarrow{AB}$ is $AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$ which is the distance between the points A and B.

Example:

If A is (3, -1, 2) and B is (1, 0, -2) find:

(a) $\overline{\mathbf{OA}}$

$$\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$$

(b) $\overline{\mathbf{AB}}$

$$\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} - \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ -4 \end{pmatrix}$$

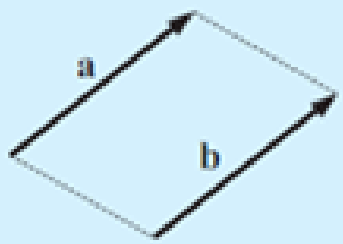
(c) $|\overline{\mathbf{AB}}|$

$$|\overrightarrow{AB}| = \sqrt{(-2)^2 + (1)^2 + (-4)^2}$$
$$= \sqrt{4+1+16}$$
$$= \sqrt{21}$$

VECTOR EQUALITY

Two vectors are **equal** if they have the same magnitude and direction.

If $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$, then $\mathbf{a} = \mathbf{b} \Leftrightarrow a_1 = b_1,\ a_2 = b_2,\ a_3 = b_3$.



If a and b do not coincide, then they are opposite sides of a parallelogram, and lie in the same plane.

Find a, b, and c if: $\begin{pmatrix} a-3 \\ 2b-1 \\ c+2 \end{pmatrix} = \begin{pmatrix} 4 \\ -b \\ 2c \end{pmatrix}$

$a-3=4 \therefore a=7$

$2b-1=-b \therefore b=\frac{1}{3}$

$c+2=2c \therefore c=2$

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14 Find k given the $\underbrace{\text{unit}}_{}$ vector: length of 1 unit

a $\begin{pmatrix} -\frac{1}{2} \\ k \\ \frac{1}{4} \end{pmatrix} = \vec{v}$

$$|\vec{v}| = \sqrt{\left(-\frac{1}{2}\right)^2 + (k)^2 + \left(\frac{1}{4}\right)^2}$$

$$= \sqrt{\frac{1}{4} + k^2 + \frac{1}{16}}$$

$$(1)^2 = \left(\sqrt{\frac{5}{16} + k^2}\right)^2$$

$$1 = \frac{5}{16} + k^2$$

$$\frac{16}{16} - \frac{5}{16} = k^2$$

$$\frac{11}{16} = k^2$$

$$\pm\frac{\sqrt{11}}{4} = k$$

15 A(−1, 3, 4), B(2, 5, −1), C(−1, 2, −2), and D(r, s, t) are four points in space.

Find r, s, and t if: **a** $\overrightarrow{AC} = \overrightarrow{BD}$ **b** $\overrightarrow{AB} = \overrightarrow{DC}$

$$\overrightarrow{AC} = \begin{pmatrix} -1-(-1) \\ 2-3 \\ -2-4 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ -6 \end{pmatrix}$$

$$\overrightarrow{BD} = \begin{pmatrix} r-2 \\ s-5 \\ t-(-1) \end{pmatrix} = \begin{pmatrix} r-2 \\ s-5 \\ t+1 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 1 \\ -6 \end{pmatrix} = \begin{pmatrix} r-2 \\ s-5 \\ t+1 \end{pmatrix}$$

$r-2=0 \therefore r=2$

$s-5=1 \therefore s=6$

$t+1=-6 \therefore t=-7$

12H – Operations with Vectors in Space

- Similar to operations with 2-D vectors

If $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ then $\mathbf{a} + \mathbf{b} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{pmatrix}$, $\mathbf{a} - \mathbf{b} = \begin{pmatrix} a_1 - b_1 \\ a_2 - b_2 \\ a_3 - b_3 \end{pmatrix}$,

and $k\mathbf{a} = \begin{pmatrix} ka_1 \\ ka_2 \\ ka_3 \end{pmatrix}$ for any scalar k.

- $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$ {commutative property} zero vector
- $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$ {associative property}
- $\mathbf{a} + \mathbf{0} = \mathbf{0} + \mathbf{a} = \mathbf{a}$ {additive identity}
- $\mathbf{a} + (-\mathbf{a}) = (-\mathbf{a}) + \mathbf{a} = \mathbf{0}$ {additive inverse}
- $\underbrace{|k\mathbf{a}|}_{\text{length of } k\mathbf{a}} = \underbrace{|k|}_{\text{modulus of } k}\;\underbrace{|\mathbf{a}|}_{\text{length of } \mathbf{a}}$ where $k\mathbf{a}$ is parallel to $\mathbf{a}$
- $k(\mathbf{a} + \mathbf{b}) = k\mathbf{a} + k\mathbf{b}$ {distributive property}

- if $\mathbf{x} + \mathbf{a} = \mathbf{b}$ then $\mathbf{x} = \mathbf{b} - \mathbf{a}$
- if $k\mathbf{x} = \mathbf{a}$ then $\mathbf{x} = \frac{1}{k}\mathbf{a}$ $(k \neq 0)$

Examples:

1. For $\vec{a} = \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix}$ find:

(a) **a** – **b**

$$\begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix} - \begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -5 \\ 6 \end{pmatrix}$$

(b) **b** + 2**a**

$$\begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix} + 2\begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 9 \end{pmatrix}$$

(c) |**a**| $= \sqrt{(1)^2 + (-2)^2 + (5)^2}$

$$= \sqrt{1+4+25}$$

$$|\vec{a}| = \sqrt{30}$$

(d) |**a** + **b**|

$$\vec{a} + \vec{b} = \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix} + \begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}$$

$$|\vec{a} + \vec{b}| = \sqrt{(1)^2 + (1)^2 + (4)^2}$$

$$= \sqrt{1+1+16}$$

$$= \sqrt{18}$$

$$= 3\sqrt{2}$$

Examples:

1. For $\mathrm{a} = \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix}$ and $\mathrm{b} = \begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix}$ find:

(e) $\mathbf{x}$ if $\frac{1}{2}\mathbf{x} = \mathbf{b}$

$$\vec{x} = 2\vec{b}$$

$$\vec{x} = 2\begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 6 \\ -2 \end{pmatrix}$$

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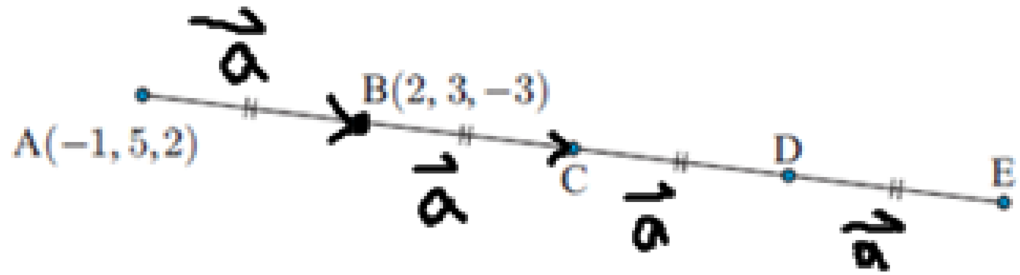
8 Find the coordinates of C, D, and E.



$$\overrightarrow{AB} = \begin{pmatrix} 3 \\ -2 \\ -5 \end{pmatrix}$$

$$C = (C_1, C_2, C_3)$$
$$= (2+3, 3+-2, -3+-5)$$
$$= (5, 1, -8)$$

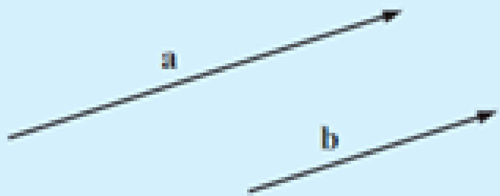
D) $(5+3, 1+(-2), -8+(-5))$
$(8, -1, -13)$

E) $(11, -3, -18)$

12I – Parallelism

If two vectors are parallel, then one is a scalar multiple of the other.

Given any non-zero vector **v** and non-zero scalar k, the vector $k\mathbf{v}$ is parallel to **v**.



- If **a** is parallel to **b**, then there exists a scalar k such that $\mathbf{a} = k\mathbf{b}$.
- If $\mathbf{a} = k\mathbf{b}$ for some scalar k, then
  - **a** is parallel to **b**, and
  - $|\mathbf{a}| = |k|\,|\mathbf{b}|$.

Example:

1. If $\mathbf{a} = \begin{pmatrix} 15 \\ 10 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ then **a** is parallel to **b**.

Find: $|\mathbf{a}|$ and $|\mathbf{b}|$

2. Find r and s given that $\mathbf{a} = \begin{pmatrix} 2 \\ -1 \\ r \end{pmatrix}$ is parallel to $\mathbf{b} = \begin{pmatrix} 8 \\ s \\ -6 \end{pmatrix}$.

Unit Vectors:

Given a non-zero vector $\mathbf{v}$, its magnitude $|\mathbf{v}|$ is a scalar quantity.

If we multiply $\mathbf{v}$ by the scalar $\frac{1}{|\mathbf{v}|}$, we obtain the parallel vector $\frac{1}{|\mathbf{v}|}\mathbf{v}$.

The length of this vector is $\left|\frac{1}{|\mathbf{v}|}\right||\mathbf{v}| = \frac{|\mathbf{v}|}{|\mathbf{v}|} = 1$, so $\frac{1}{|\mathbf{v}|}\mathbf{v}$ is a unit vector in the direction of $\mathbf{v}$.

- A unit vector in the direction of $\mathbf{v}$ is $\frac{1}{|\mathbf{v}|}\mathbf{v}$.
- A vector $\mathbf{b}$ of length k in the same direction as $\mathbf{a}$ is $\mathbf{b} = \frac{k}{|\mathbf{a}|}\mathbf{a}$.
- A vector $\mathbf{b}$ of length k which is *parallel to* $\mathbf{a}$ could be $\mathbf{b} = \pm\frac{k}{|\mathbf{a}|}\mathbf{a}$.

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6 Find the unit vector in the direction of:

a $\mathbf{i} + 2\mathbf{j}$ **b** $2\mathbf{i} - 3\mathbf{k}$ **c** $2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$

10 Find a vector **b** in:

a the same direction as $\begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}$ and with length 6 units