24 C – The Standard Normal Distribution (*Z*-Distribution)

Every normal *X*-distribution can be transformed into the standard normal distribution, also called the *Z*-distribution using the transformation

$$z = \frac{x - \mu}{\sigma}$$

No matter what the values of μ and σ are of the *X*-distribution, the *Z*-distribution will always have mean 0 and standard deviation 1.

$$Z \sim N(0, 1)$$

Read pg. 640 for the *Z*-transformation…

The **probability density function** for the *z*-distribution is:

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \qquad -\infty < z < \infty$$

If *x* is an observation from a normal distribution with mean μ and standard deviation σ, the ***z*-score** of *x* is the number of standard deviations *x* is from the mean.

$$z = \frac{x-\mu}{\sigma}$$

For example, if $z = 1.34$, then *x* is 1.34 standard deviations to the right of the mean.
If $z = -0.31$, then *x* is 0.31 standard deviations to the left of the mean.

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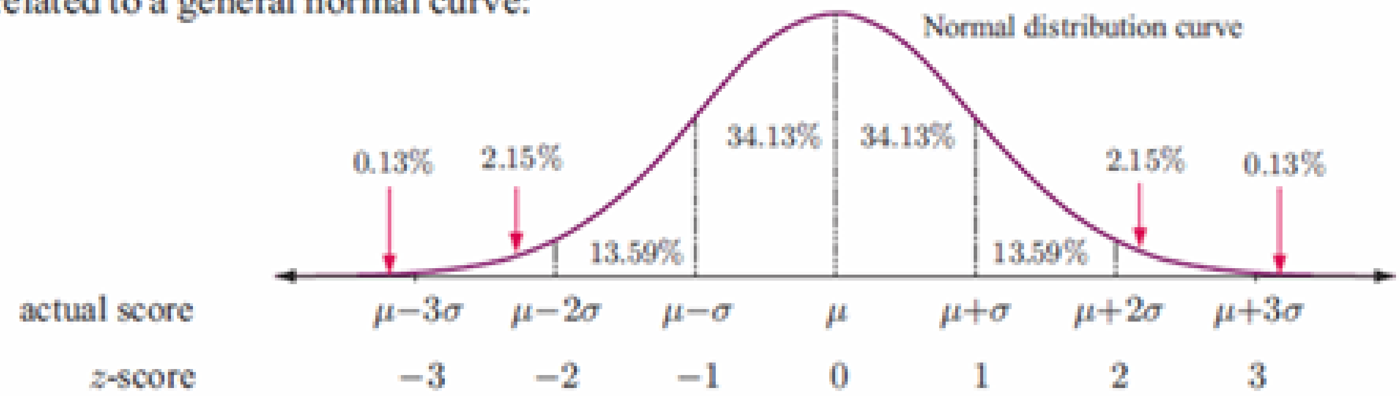
This diagram shows how the z-score is related to a general normal curve:

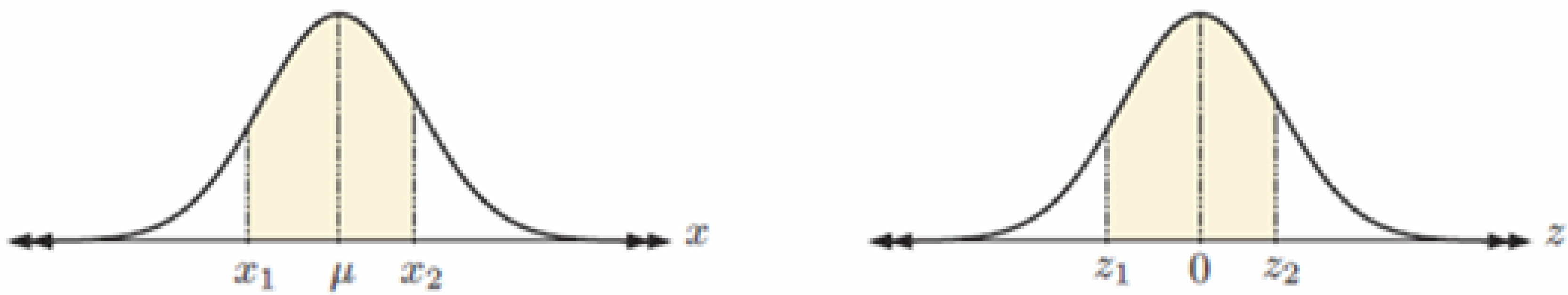


This means that for $X \sim \mathrm{N}(\mu, \sigma^2)$ and $Z \sim \mathrm{N}(0, 1^2)$ and for any $x_1, x_2 \in \mathbb{R}$, $x_1 < x_2$, with corresponding z-values $z_1 = \dfrac{x_1 - \mu}{\sigma}$ and $z_2 = \dfrac{x_2 - \mu}{\sigma}$, then:

- $\mathrm{P}(X \geqslant x_1) = \mathrm{P}(Z \geqslant z_1)$

- $\mathrm{P}(X \leqslant x_1) = \mathrm{P}(Z \leqslant z_1)$

- $\mathrm{P}(x_1 \leqslant X \leqslant x_2) = \mathrm{P}(z_1 \leqslant Z \leqslant z_2)$



z-scores are useful when comparing two populations with different means and standard deviations (as long as both distributions are approximately normal).

Example:

Pg. 642

1 The table shows Emma's midyear exam results. The exam results for each subject are normally distributed with the mean μ and standard deviation σ shown in the table.

a Find the z-score for each of Emma's subjects.

b Arrange Emma's subjects from 'best' to 'worst' in terms of the z-scores.

Subject	*Emma's score*	μ	σ
English	48	40	4.4
Mandarin	81	60	9
Geography	84	55	18
Biology	68	50	20
Maths	84	50	15

$z = \frac{x - \mu}{\sigma}$

English:

$z = \frac{48-40}{4.4}$

$z = 1.82$

Mandarin

$z = \frac{81-60}{9}$

$= 2.33$

Geo

$z = \frac{84-55}{18}$

$z = 1.61$

Bio

$z = \frac{68-50}{20}$

$z = 0.9$

Maths

$z = \frac{84-50}{15}$

$z = 2.27$

Best sub: Mandarin

Worst: Bio

- Mandarin
- Maths
- English
- Geo
- Bio

Pg. 643

5 If Z is the standard normal distribution, find the following probabilities using technology. In each case sketch the regions.

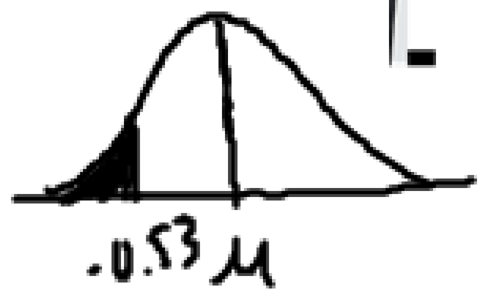
a $P(-0.86 \leqslant Z \leqslant 0.32)$

d $P(Z \leqslant -0.53)$

g $P(Z > 4)$

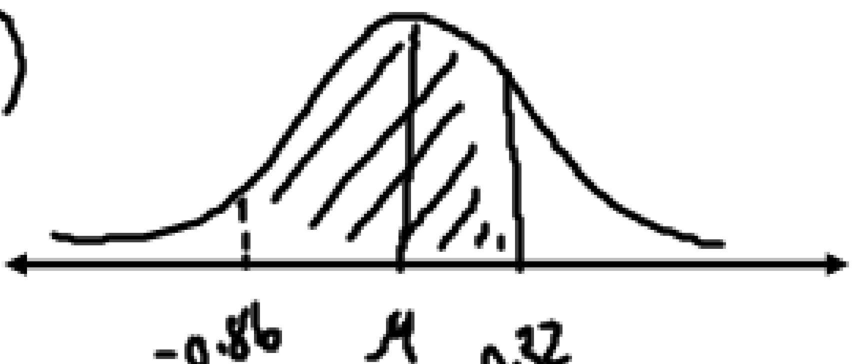
$\mu = 0 \quad \sigma = 1$

D)




```
normalcdf(-1E99
          .2980559431
```

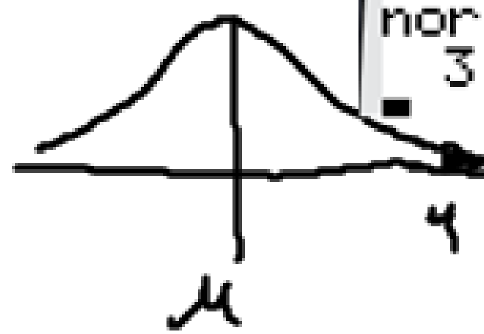
A)



Normalcdf

```
normalcdf(-.86,
          .430621297
```

g)




```
normalcdf(4,1*1
       3.168603459E-5
```

0.00003168

24D – Quantiles or *k*-values

When finding quantiles, we are given a probability and are asked to calculate the corresponding measurement.

For example, the masses of Oreo cookies are normally distributed with a mean of 40 g and a standard deviation of 2 g.

Find the value of k such that approximately 16% of the cookies are below k g.

We need to find k such that $P(X \leq k) = 0.16$.

Use the **inverse normal function** on the calculator:

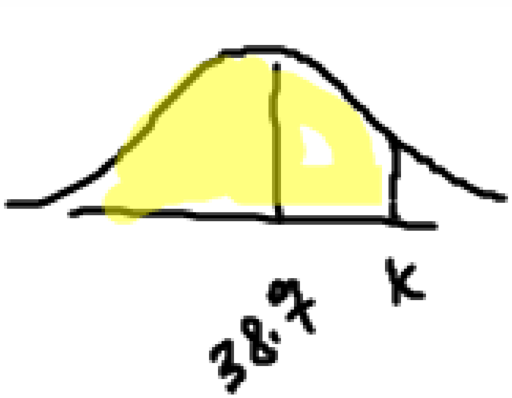
[2nd] [vars] invNorm( )

Pg. 645

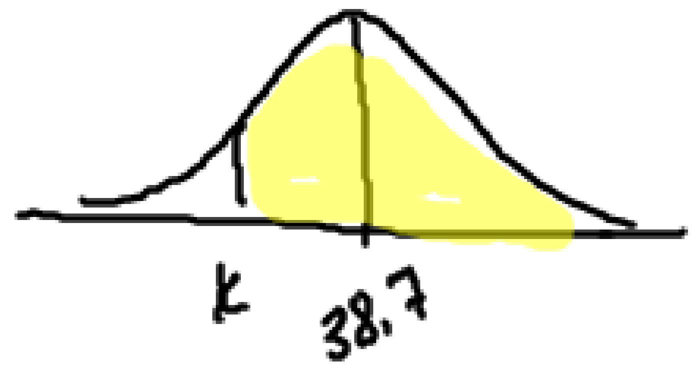
mean ← 8.2 st.dev

3 Suppose $X \sim N(38.7, 8.2^2)$. Illustrate with a sketch and find k such that:

a $P(X \leqslant k) = 0.9$ **b** $P(X \geqslant k) = 0.8$




```
invNorm(.9,38.7▸
         49.20872285
```



invNorm( ) → only for $(X \leq x)$

$P(X \geq k) = 1 - P(X < k)$

$P(X < k) = 1 - P(X \geq k)$

$= 1 - 0.8$

$= 0.2$

```
invNorm(.2,38.7▸
         31.79870589
```

** If we are trying to find an unknown mean μ or standard deviation σ, then we **always** need to convert to z-scores.

Pg. 646

1 The IQs of students at school are normally distributed with a standard deviation of 15. If 20% of students have an IQ higher than 125, find the mean IQ of students at school.

let X be the IQ score of a student, then the mean IQ score of the students is μ

→ we can't use invNorm yet because we don't know μ

need to convert to z scores

Pg. 647

5 **a** Find the mean and standard deviation of a normally distributed random variable X, given that $P(X \geqslant 80) = 0.1$ and $P(X \leqslant 30) = 0.15$.

b In a Mathematics examination it was found that 10% of the students scored at least 80, and no more than 15% scored under 30. Assuming the scores are normally distributed, what proportion of students scored more than 50?