18F – Integrating $f(ax+b)$

Notice that: $$\frac{d}{dx}\left(\frac{1}{a}e^{ax+b}\right)=\frac{1}{a}e^{ax+b}\times a=e^{ax+b}$$

$$\int e^{ax+b}\,dx=\frac{1}{a}e^{ax+b}+c \quad \text{for } a\neq 0$$

$\int$

If $n \neq -1$,

$$\frac{d}{dx}\left(\frac{1}{a(n+1)}(ax+b)^{n+1}\right) = \frac{1}{a(n+1)}(n+1)(ax+b)^{n} \times a$$

$$= (ax+b)^{n}$$

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$$\int (ax+b)^{n}\, dx = \frac{1}{a}\,\frac{(ax+b)^{n+1}}{(n+1)} + c \quad \text{for } n \neq -1$$

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For a, b constants with $a \neq 0$

Function	*Integral*
$\int e^{ax+b}\,dx$	$\frac{1}{a}e^{ax+b} + c$
$\int (ax+b)^n\,dx,\ n \neq -1$	$\frac{1}{a}\frac{(ax+b)^{n+1}}{n+1} + c$
$\int \frac{1}{ax+b}\,dx$	$\frac{1}{a}\ln\lvert ax+b\rvert + c$
$\int \cos(ax+b)\,dx$	$\frac{1}{a}\sin(ax+b) + c$
$\int \sin(ax+b)\,dx$	$-\frac{1}{a}\cos(ax+b) + c$

Examples:

Find

(a) $\int (3x+1)^5 dx$

$= \frac{1}{6}(3x+1)^6\left(\frac{1}{3}\right)+C$

$= \frac{1}{18}(3x+1)^6+C = \frac{(3x+1)^6}{18}+C$

check: $y = \frac{1}{18}(3x+1)^6$

$y' = \frac{1}{18}\left(6(3x+1)^5 \cdot 3\right)$

(b) $\int \frac{1}{\sqrt{2-3x}} dx = \int (2-3x)^{-\frac{1}{2}} dx$

$= -\frac{1}{3}\left(\frac{1}{\frac{1}{2}}(2-3x)^{\frac{1}{2}}\right)+C$

$= \frac{-1}{3/2}(2-3x)^{1/2}+C$

$= -\frac{2}{3}\sqrt{2-3x}+C$

(c) $\int \left[3e^{2x+1} + (2x-1)^2\right] dx$

$$= \int 3e^{2x+1}\,dx + \int (2x-1)^2\,dx$$

$$= 3\int e^{2x+1}\,dx + \int (2x-1)^2\,dx$$

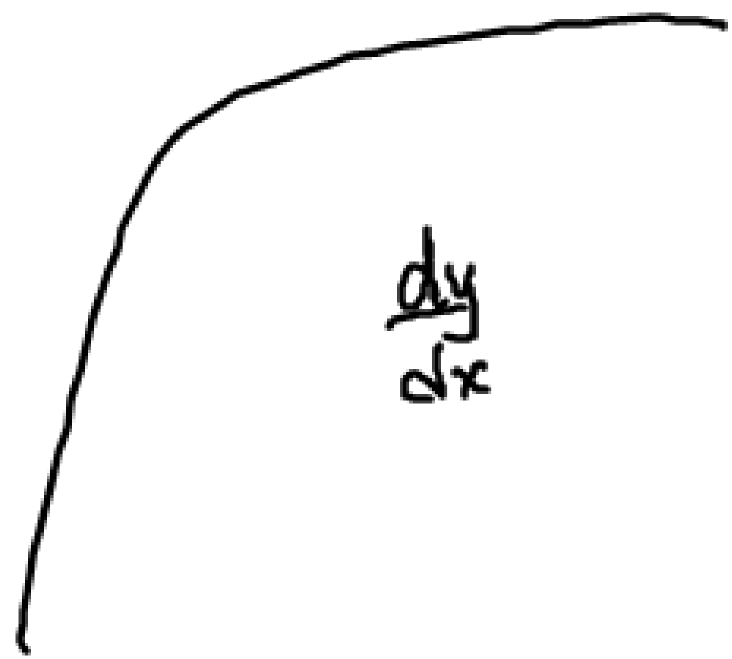
$$= 3\left(\frac{1}{2}e^{2x+1}\right) + \frac{1}{2}(2x-1)^3 \cdot \left(\frac{1}{3}\right)$$

$$= \frac{3}{2}e^{2x+1} + \frac{1}{6}(2x-1)^3 + C$$

(d) $\int \frac{2}{1-3x}\,dx = 2\int \frac{1}{1-3x}\,dx$

$$= 2\left(\frac{1}{-3}\ln|1-3x|\right) + C$$

$$= -\frac{2}{3}\ln|1-3x| + C$$



(e) Integrate with respect to x:

$3\cos(2x)+\sin(5x+\pi)$ $\Rightarrow \int (3\cos(2x)+\sin(5x+\pi))dx$

$$= \int 3\cos(2x)dx + \int \sin(5x+\pi)dx$$

$$= 3\int \cos(2x)dx + \int \sin(5x+\pi)dx$$

$$= 3\left(\frac{1}{2}\sin(2x)\right) + \frac{-1}{5}\cos(5x+\pi) + C$$

$$= \frac{3}{2}\sin(2x) - \frac{1}{5}\cos(5x+\pi) + C$$

(f) Integrate $(2-\sin x)^2$ with respect to x.

Remember identities!

$\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = \underbrace{1 - 2\sin^2 x}$

$$\int (2-\sin x)^2 dx = \int (2-\sin x)(2-\sin x)\,dx$$

$$= \int (4 - 4\sin x + \sin^2 x)\,dx$$

$\cos 2x = 1 - 2\sin^2 x$

$\frac{\cos 2x - 1}{-2} = \frac{-2\sin^2 x}{-2}$

$-\frac{1}{2}\cos(2x) + \frac{1}{2} = \sin^2 x$

$$= \int \left(4 - 4\sin x + \frac{1}{2} - \frac{1}{2}\cos(2x)\right)dx$$

$$= \int \left(\frac{9}{2} - 4\sin x - \frac{1}{2}\cos(2x)\right)dx$$

$$= \frac{9}{2}x - 4(-\cos x) - \frac{1}{2}\left(\frac{1}{2}\sin(2x)\right) + C$$

12 Show that $(\sin x + \cos x)^2 = 1 + \sin 2x$ and hence determine $\int (\sin x + \cos x)^2\, dx$.

$$(\sin x + \cos x)(\sin x + \cos x)$$
$$= \sin^2 x + 2\sin x\cos x + \cos^2 x$$
$$= \underbrace{\sin^2 x + \cos^2 x} + 2\sin x\cos x$$
$$= 1 + 2\sin x\cos x$$
$$= 1 + \sin 2x$$

$$\int (\sin x + \cos x)^2\, dx$$
$$= \int (1 + \sin 2x)\, dx$$
$$= x + \left(-\frac{1}{2}\cos(2x)\right) + C$$
$$= x - \frac{1}{2}\cos(2x) + C$$

EXERCISE 18F

1 Find:

a $\int (2x+5)^3\,dx$
b $\int \frac{1}{(3-2x)^2}\,dx$
c $\int \frac{4}{(2x-1)^4}\,dx$

d $\int (4x-3)^7\,dx$
e $\int \sqrt{3x-4}\,dx$
f $\int \frac{10}{\sqrt{1-5x}}\,dx$

g $\int 3(1-x)^4\,dx$
h $\int \frac{4}{\sqrt{3-4x}}\,dx$

2 Integrate with respect to x:

a $\sin(3x)$
b $2\cos(-4x)+1$
c $3\cos\left(\frac{x}{2}\right)$

d $3\sin(2x)-e^{-x}$
e $2\sin\left(2x+\frac{\pi}{6}\right)$
f $-3\cos\left(\frac{\pi}{4}-x\right)$

g $\cos(2x)+\sin(2x)$
h $2\sin(3x)+5\cos(4x)$
i $\frac{1}{2}\cos(8x)-3\sin x$

3 **a** Find $y=f(x)$ given $\frac{dy}{dx}=\sqrt{2x-7}$ and that $y=11$ when $x=8$.

b The function $f(x)$ has gradient function $f'(x)=\frac{4}{\sqrt{1-x}}$, and the curve $y=f(x)$ passes through the point $(-3, -11)$.
Find the point on the graph of $y=f(x)$ with x-coordinate -8.

4 Using the identities $\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos(2x)$ and $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos(2x)$ to help you, integrate with respect to x:

a $\cos^2 x$ **b** $\sin^2 x$ **c** $1 + \cos^2(2x)$

d $3 - \sin^2(3x)$ **e** $\frac{1}{2}\cos^2(4x)$ **f** $(1 + \cos x)^2$

5 Find:

a $\int 3(2x-1)^2\,dx$ **b** $\int (x^2 - x)^2\,dx$ **c** $\int (1-3x)^3\,dx$

d $\int (1-x^2)^2\,dx$ **e** $\int 4\sqrt{5-x}\,dx$ **f** $\int (x^2+1)^3\,dx$

6 Find:

a $\int (2e^x + 5e^{2x})\, dx$ **b** $\int (3e^{5x-2})\, dx$ **c** $\int (e^{7-3x})\, dx$

d $\int \frac{1}{2x-1}\, dx$ **e** $\int \frac{5}{1-3x}\, dx$ **f** $\int \left(e^{-x} - \frac{4}{2x+1}\right) dx$

g $\int (e^x + e^{-x})^2\, dx$ **h** $\int (e^{-x} + 2)^2\, dx$ **i** $\int \left(x - \frac{5}{1-x}\right) dx$

7 Find y given that:

a $\frac{dy}{dx} = (1 - e^x)^2$ **b** $\frac{dy}{dx} = 1 - 2x + \frac{3}{x+2}$ **c** $\frac{dy}{dx} = e^{-2x} + \frac{4}{2x-1}$

8 To find $\int \frac{1}{4x}\, dx$, where $x > 0$, Tracy's answer was $\int \frac{1}{4x}\, dx = \frac{1}{4}\ln(4x) + c, \ x > 0$

and Nadine's answer was $\int \frac{1}{4x}\, dx = \frac{1}{4}\int \frac{1}{x}\, dx = \frac{1}{4}\ln x + c, \ x > 0.$

Which of them has found the correct answer? Prove your statement.

9 Suppose $f'(x) = p\sin\left(\frac{1}{2}x\right)$ where p is a constant. $f(0) = 1$ and $f(2\pi) = 0$. Find p and hence $f(x)$.

10 Consider a function g such that $g''(x) = -\sin 2x$.
Show that the gradients of the tangents to $y = g(x)$ when $x = \pi$ and $x = -\pi$ are equal.

11 **a** Find $f(x)$ given $f'(x) = 2e^{-2x}$ and $f(0) = 3$.

b Find $f(x)$ given $f'(x) = 2x - \frac{2}{1-x}$ and $f(-1) = 3$.

c A curve has gradient function $\sqrt{x} + \frac{1}{2}e^{-4x}$ and passes through $(1, 0)$. Find the equation of the function.

12 Show that $(\sin x + \cos x)^2 = 1 + \sin 2x$ and hence determine $\int (\sin x + \cos x)^2\, dx$.

13 Show that $(\cos x + 1)^2 = \frac{1}{2}\cos 2x + 2\cos x + \frac{3}{2}$ and hence determine $\int (\cos x + 1)^2\, dx$.