

18F – Integrating $f(ax+b)$

Notice that: $\frac{d}{dx}\left(\frac{1}{a}e^{ax+b}\right) = \frac{1}{a}e^{ax+b} \times a = e^{ax+b}$

$$\int e^{ax+b} dx = \frac{1}{a} e^{ax+b} + c \quad \text{for } a \neq 0$$

If $n \neq -1$,

$$\begin{aligned}\frac{d}{dx} \left(\frac{1}{a(n+1)} (ax+b)^{n+1} \right) &= \frac{1}{a(n+1)} (n+1)(ax+b)^n \times a \\ &= (ax+b)^n\end{aligned}$$

$$\int (ax+b)^n dx = \frac{1}{a} \frac{(ax+b)^{n+1}}{(n+1)} + c \quad \text{for } n \neq -1$$

$$\frac{1}{3} \frac{(3x+1)^6}{6}$$

For a, b constants with $a \neq 0$

Function	Integral
$\int e^{ax+b} dx$	$\frac{1}{a} e^{ax+b} + c$
$\int (ax+b)^n dx \neq -1$	$\frac{1}{a} \frac{(ax+b)^{n+1}}{n+1} + c$
$\frac{1}{ax+b}$	$\frac{1}{a} \ln ax+b + c$
$\cos(ax+b)$	$\frac{1}{a} \sin(ax+b) + c$
$\sin(ax+b)$	$-\frac{1}{a} \cos(ax+b) + c$

$$\int e^x dx$$

$$\int \frac{1}{x} dx$$

$$\int \cos x dx$$

$$\int \sin x dx$$

Examples:

$$\int x^5 dx = \frac{1}{6} x^6 + c$$

$ax+b$

$b-ax$

Find

(a) $\int (3x+1)^5 dx$

$$= \frac{1}{6} (3x+1)^6 \left(\frac{1}{3}\right) + C$$

$$= \frac{1}{18} (3x+1)^6 + C$$

Check:

$$\frac{d}{dx} \left(\frac{1}{18} (3x+1)^6 \right)$$

$$\frac{1}{18} (6 (3x+1)^5 (3))$$

← chain
Rule

(b) $\int \frac{1}{\sqrt{2-3x}} dx$

$$= \int (2-3x)^{-\frac{1}{2}} dx$$

$$= \frac{1}{-\frac{1}{2}} \frac{(2-3x)^{\frac{1}{2}}}{-\frac{3}{2}} + C$$

$$= \frac{-2}{-3} (2-3x)^{\frac{1}{2}} + C$$

$$\begin{aligned}
\text{(c)} \quad & \int (3e^{2x+1} + (2x-1)^2) dx \\
&= \int 3e^{2x+1} dx + \int (2x-1)^2 dx \\
&= 3 \int e^{2x+1} dx + \int (2x-1)^2 dx \\
&= 3 \left(\frac{1}{2} e^{2x+1} \right) + \frac{1}{3} (2x-1)^3 \left(\frac{1}{2} \right) + C \\
&= \frac{3}{2} e^{2x+1} + \frac{1}{6} (2x-1)^3 + C.
\end{aligned}$$

$$\begin{aligned}
\text{(d)} \quad & \int \frac{2}{1-3x} dx \\
&= 2 \int \frac{1}{1-3x} dx \\
&= 2 \left(\frac{1}{-3} \ln|1-3x| + C \right) \\
&= -\frac{2}{3} \ln|1-3x| + C
\end{aligned}$$

(e) Integrate with respect to x :

$$3 \cos(2x) + \sin(5x + \pi)$$

$$\int (3 \cos(2x) + \sin(5x + \pi)) dx$$

$$= \int 3 \cos(2x) dx + \int \sin(5x + \pi) dx$$

$$= 3 \int \cos(2x) dx + \int \sin(5x + \pi) dx$$

$$= 3 \left(\frac{1}{2} \sin(2x) \right) + \left(-\frac{1}{5} \cos(5x + \pi) \right) + C$$

$$= \frac{3}{2} \sin(2x) - \frac{1}{5} \cos(5x + \pi) + C$$

(f) Integrate $(2 - \sin x)^2$ with respect to x .

Remember identities!

$$\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$$

$$\int (2 - \sin x)^2 dx$$

$$= \int (2 - \sin x)(2 - \sin x) dx$$

$$= \int (4 - 4\sin x + \sin^2 x) dx$$

$$= \int \left(4 - 4\sin x + \frac{1}{2} - \frac{1}{2}\cos 2x \right) dx$$

$$= \int \left(\frac{9}{2} - 4\sin x - \frac{1}{2}\cos 2x \right) dx$$

$$\cos 2x = 1 - 2\sin^2 x$$

$$2\sin^2 x = 1 - \cos 2x$$

$$\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$$

$$= \frac{9}{2}x - 4(-\cos x) - \frac{1}{2} \left(\frac{1}{2}\sin(2x) \right) + C$$

$$= \frac{9}{2}x + 4\cos x - \frac{1}{4}\sin(2x) + C$$

A)

B)

12 Show that $(\sin x + \cos x)^2 = 1 + \sin 2x$ and hence determine $\int (\sin x + \cos x)^2 dx$.

$$\begin{aligned}
 (\sin x + \cos x)^2 &= (\sin x + \cos x)(\sin x + \cos x) \\
 &= \sin^2 x + 2\sin x \cos x + \cos^2 x \\
 &= \sin^2 x + \cos^2 x + 2\sin x \cos x \\
 &= 1 + \sin 2x
 \end{aligned}$$

$$\begin{aligned}
 \int (\sin x + \cos x)^2 dx &= \int (1 + \sin 2x) dx \\
 &= x + \left(-\frac{1}{2} \cos 2x\right) + C \\
 &= x - \frac{1}{2} \cos 2x + C
 \end{aligned}$$

1 Find:

a $\int (2x + 5)^3 dx$

b $\int \frac{1}{(3 - 2x)^2} dx$

c $\int \frac{4}{(2x - 1)^4} dx$

d $\int (4x - 3)^7 dx$

e $\int \sqrt{3x - 4} dx$

f $\int \frac{10}{\sqrt{1 - 5x}} dx$

g $\int 3(1 - x)^4 dx$

h $\int \frac{4}{\sqrt{3 - 4x}} dx$

2 Integrate with respect to x :

a $\sin(3x)$

b $2 \cos(-4x) + 1$

c $3 \cos\left(\frac{\pi}{2}\right)$

d $3 \sin(2x) - e^{-x}$

e $2 \sin\left(2x + \frac{\pi}{6}\right)$

f $-3 \cos\left(\frac{\pi}{4} - x\right)$

g $\cos(2x) + \sin(2x)$

h $2 \sin(3x) + 5 \cos(4x)$

i $\frac{1}{2} \cos(8x) - 3 \sin x$

3 a Find $y = f(x)$ given $\frac{dy}{dx} = \sqrt{2x - 7}$ and that $y = 11$ when $x = 8$.

b The function $f(x)$ has gradient function $f'(x) = \frac{4}{\sqrt{1 - x}}$, and the curve $y = f(x)$ passes through the point $(-3, -11)$.

Find the point on the graph of $y = f(x)$ with x -coordinate -8 .

4 Using the identities $\cos^2 x = \frac{1}{2} + \frac{1}{2} \cos(2x)$ and $\sin^2 x = \frac{1}{2} - \frac{1}{2} \cos(2x)$ to help you, integrate with respect to x :

a $\cos^2 x$

b $\sin^2 x$

c $1 + \cos^2(2x)$

d $3 - \sin^2(3x)$

e $\frac{1}{2} \cos^2(4x)$

f $(1 + \cos x)^2$

5 Find:

a $\int 3(2x - 1)^2 dx$

b $\int (x^2 - x)^2 dx$

c $\int (1 - 3x)^3 dx$

d $\int (1 - x^2)^2 dx$

e $\int 4\sqrt{5 - x} dx$

f $\int (x^2 + 1)^3 dx$

6 Find:

a $\int (2e^x + 5e^{2x}) dx$

b $\int (3e^{5x-2}) dx$

c $\int (e^{7-3x}) dx$

d $\int \frac{1}{2x-1} dx$

e $\int \frac{5}{1-3x} dx$

f $\int \left(e^{-x} - \frac{4}{2x+1} \right) dx$

g $\int (e^x + e^{-x})^2 dx$

h $\int (e^{-x} + 2)^2 dx$

i $\int \left(x - \frac{5}{1-x} \right) dx$

7 Find y given that:

a $\frac{dy}{dx} = (1 - e^x)^2$

b $\frac{dy}{dx} = 1 - 2x + \frac{3}{x+2}$

c $\frac{dy}{dx} = e^{-2x} + \frac{4}{2x-1}$

8 To find $\int \frac{1}{4x} dx$, where $x > 0$, Tracy's answer was $\int \frac{1}{4x} dx = \frac{1}{4} \ln(4x) + c$, $x > 0$

and Nadine's answer was $\int \frac{1}{4x} dx = \frac{1}{4} \int \frac{1}{x} dx = \frac{1}{4} \ln x + c$, $x > 0$.

Which of them has found the correct answer? Prove your statement.

9 Suppose $f'(x) = p \sin\left(\frac{1}{2}x\right)$ where p is a constant. $f(0) = 1$ and $f(2\pi) = 0$. Find p and hence $f(x)$.

10 Consider a function g such that $g''(x) = -\sin 2x$.

Show that the gradients of the tangents to $y = g(x)$ when $x = \pi$ and $x = -\pi$ are equal.

11 a Find $f(x)$ given $f'(x) = 2e^{-2x}$ and $f(0) = 3$.

b Find $f(x)$ given $f'(x) = 2x - \frac{2}{1-x}$ and $f(-1) = 3$.

c A curve has gradient function $\sqrt{x} + \frac{1}{2}e^{-4x}$ and passes through $(1, 0)$. Find the equation of the function.

12 Show that $(\sin x + \cos x)^2 = 1 + \sin 2x$ and hence determine $\int (\sin x + \cos x)^2 dx$.

13 Show that $(\cos x + 1)^2 = \frac{1}{2} \cos 2x + 2 \cos x + \frac{3}{2}$ and hence determine $\int (\cos x + 1)^2 dx$.