18D – Integration

The antiderivative of x^2 is $\frac{1}{3}x^3$.

Any function of the form $\frac{1}{3}x^3 + c$, where c is a constant, has derivative x^2.

The **indefinite integral** or **integral** of x^2 is $\frac{1}{3}x^3 + c$.

$$\int x^2 dx = \frac{1}{3}x^3 + c$$

"the integral of x^2 with respect to x is $\frac{1}{3}x^3 + c$, where c is a constant"

Indefinite integration – not being applied to a particular interval.

If $F'(x) = f(x)$ then $\int f(x)\,dx = F(x) + c$.

Examples:

Find the derivative, then write an expression for the integral:

(a) $F(x) = x^6$

$F'(x) = 6x^5$

$\int 6x^5\,dx = x^6 + C$

(b) $F(x) = x^{\frac{1}{3}}$

$F'(x) = \frac{1}{3}x^{-\frac{2}{3}}$

$\int \frac{1}{3}x^{-\frac{2}{3}}\,dx = x^{\frac{1}{3}} + C$

don't forget :)

Useful Rules:

Any constant may be written in front of the integral sign:

$$\int kf(x)dx = k\int f(x)dx, \quad k \text{ is a constant}$$

$f(x)=3x^4$

$f'(x)=3(4x^3)$

$=12x^3$

The integral of a sum is the sum of the separate integrals:

$$\int [f(x)+g(x)]dx = \int f(x)dx + \int g(x)dx$$

$f(x)= x^4+x^5$

$f'(x) = 4x^3+5x^4$

Examples:

1. If $y = x^6 + 2x^5$, find $\frac{dy}{dx}$. Hence find $\int (3x^5 + 5x^4)dx$.

$$\frac{dy}{dx} = 6x^5 + 10x^4 \longrightarrow \int (6x^5 + 10x^4)dx = x^6 + 2x^5 + c$$

$$\int 2(3x^5 + 5x^4)dx = x^6 + 2x^5 + c$$

$$\frac{2\int (3x^5 + 5x^4)dx}{2} = \frac{x^6}{2} + \frac{2x^5}{2} + \frac{c}{2}$$

$$\int (3x^5 + 5x^4)dx = \frac{1}{2}x^6 + x^5 + c$$

$$\int (3x^5 + 5x^4)dx$$

$$= 3\left(\frac{1}{6}x^6\right) + 5\left(\frac{1}{5}x^5\right) + c$$

$$= \frac{1}{2}x^6 + x^5 + c$$

2. If $y = e^{-3x}$, find $\frac{dy}{dx}$. Hence, find $\int e^{-3x}\,dx$.

$\frac{dy}{dx} = e^{-3x}(-3)$

$\frac{dy}{dx} = -3e^{-3x}$

$\int (-3e^{-3x})\,dx = e^{-3x} + C$

$-3\int e^{-3x}\,dx = e^{-3x} + C$

$\int e^{-3x}\,dx = -\frac{1}{3}e^{-3x} + C$

3. If $y=\sqrt{3x+2}$, find $\dfrac{dy}{dx}$. Hence find $\displaystyle\int \frac{1}{\sqrt{3x+2}}\,dx$

$y=(3x+2)^{\frac{1}{2}}$ chainRule

$\dfrac{dy}{dx}=\dfrac{1}{2}(3x+2)^{-\frac{1}{2}}(3)$

$\dfrac{dy}{dx}=\dfrac{3}{2\sqrt{3x+2}}$

$\displaystyle\int \frac{3}{2\sqrt{3x+2}}\,dx=\sqrt{3x+2}+c$

$\displaystyle\frac{3}{2}\int \frac{1}{\sqrt{3x+2}}\,dx=\sqrt{3x+2}+c$

$\displaystyle\int \frac{1}{\sqrt{3x+2}}\,dx=\frac{2}{3}\sqrt{3x+2}+C$

18E – Rules for Integration

Pg. 457 – Review of rules for differentiation:

Function	*Derivative*	*Name*
c, a constant	0	
$mx + c$, m and c are constants	m	
x^n	nx^{n-1}	**power rule**
$cu(x)$	$cu'(x)$	
$u(x) + v(x)$	$u'(x) + v'(x)$	**addition rule**
$u(x)v(x)$	$u'(x)v(x) + u(x)v'(x)$	**product rule**
$\frac{u(x)}{v(x)}$	$\frac{u'(x)v(x) - u(x)v'(x)}{[v(x)]^2}$	**quotient rule**
$y = f(u) \quad$ where $\quad u = u(x)$	$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$	**chain rule**
e^x	e^x	

Rules for Integration:

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For a constant k, $\frac{d}{dx}(kx+c)=k$

$$\frac{d}{dx}(e^x+c)=e^x$$

$$\int k\,dx = kx + c$$

kx^0

$$\int e^x\,dx = e^x + c$$

If $n \neq -1$, $\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}+c\right)=\frac{(n+1)x^n}{n+1}=x^n$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + c, \quad n \neq -1$$

If $x > 0$, $\frac{d}{dx}(\ln x + c) = \frac{1}{x}$

If $x < 0$, $\frac{d}{dx}(\ln(-x) + c) = \frac{-1}{-x} = \frac{1}{x}$

$$\int \frac{1}{x}\, dx = \ln|x| + c$$

$$\frac{d}{dx}(\sin x + c) = \cos x$$

$$\int \cos x\, dx = \sin x + c$$

$$\frac{d}{dx}(-\cos x + c) = \sin x$$

$$\int \sin x\, dx = -\cos x + c$$

Examples

(a) Find: $\int (x^4 - 3x^2 + \sqrt{x} + 7)\,dx$ $= \frac{1}{5}x^5 - 3\left(\frac{1}{3}x^3\right) + \frac{2}{3}x^{\frac{3}{2}} + 7x + C$

$x^{\frac{1}{2}}$

don't forget

$= \frac{1}{5}x^5 - x^3 + \frac{2}{3}x^{3/2} + 7x + C$

(b) Integrate with respect to x: $3\cos x + 2\sin x$

$\int (3\cos x + 2\sin x)\,dx = 3\sin x + 2(-\cos x) + C$

$= 3\sin x - 2\cos x + C$

(c) Integrate with respect to x: $2e^x - \frac{1}{x}$

$$\int\left(2e^x - \frac{1}{x}\right)dx = 2e^x - \ln|x| + C$$

(d) Find: $\int\left(\frac{x-3}{\sqrt{x}}\right)^2 dx$

$$= \int\left(\frac{x-3}{\sqrt{x}}\right)\left(\frac{x-3}{\sqrt{x}}\right)dx$$

$$= \int\left(\frac{x^2-6x+9}{x}\right)dx$$

$$= \int\left(x - 6 + \frac{9}{x}\right)dx$$

$$= \frac{1}{2}x^2 - 6x + 9\ln|x| + C$$

(e) Find $f(x)$ given that $f'(x) = x^3 + 3x^2 + 1$ and $f(0) = 5$.

$$f(x) = \int (x^3 + 3x^2 + 1)\,dx$$

$$f(x) = \frac{1}{4}x^4 + 3\left(\frac{1}{3}x^3\right) + x + C$$

$$f(x) = \frac{1}{4}x^4 + x^3 + x + C$$

$$5 = \frac{1}{4}(0)^4 + 0^3 + 0 + C$$

$$5 = C$$

$$f(x) = \frac{1}{4}x^4 + x^3 + x + 5$$

HW: ch 18D &E