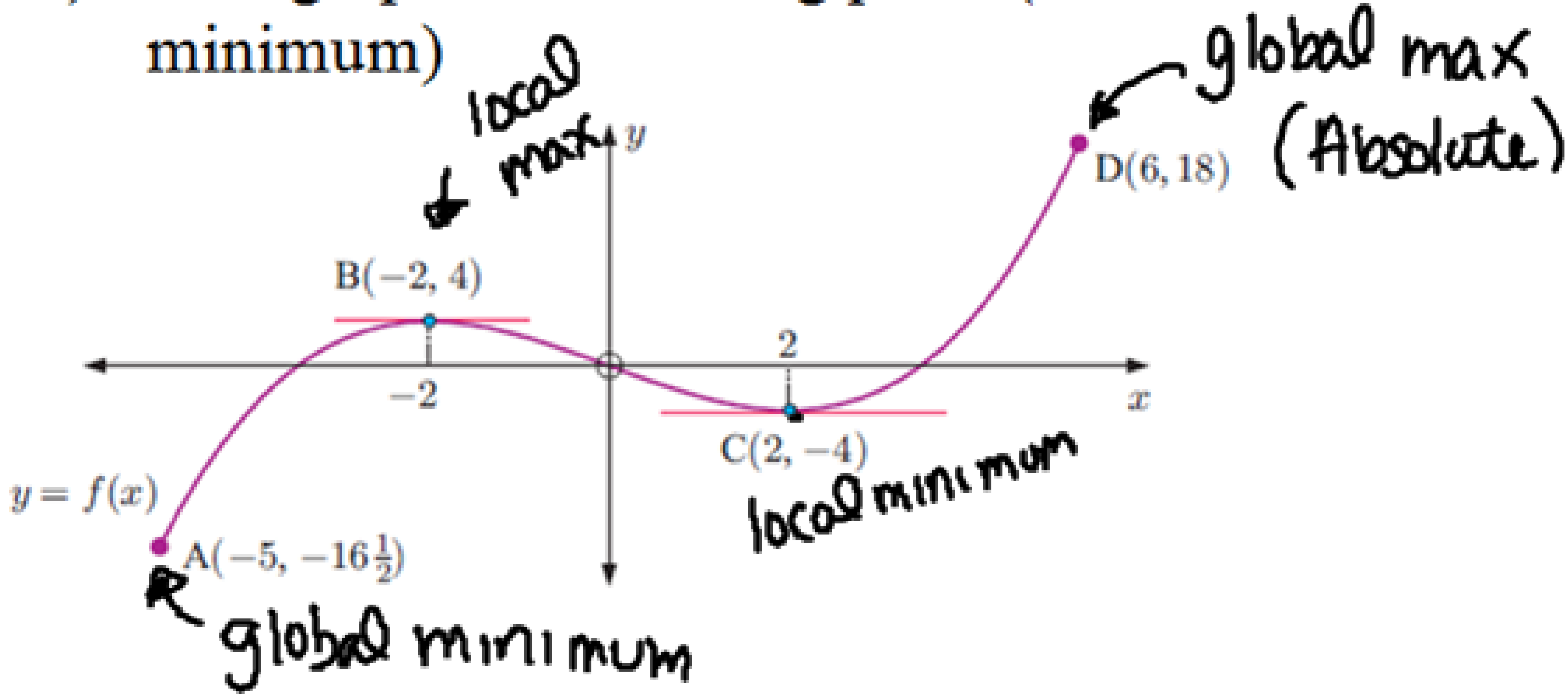
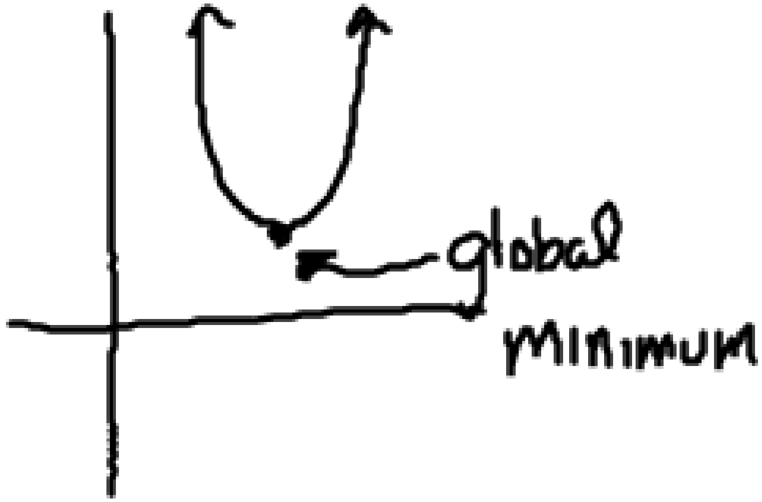
16C – Stationary Points

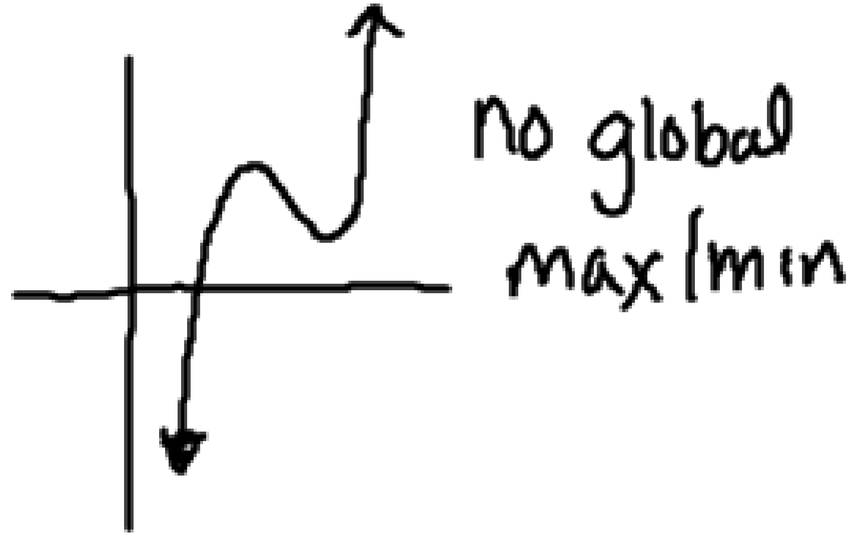
A stationary point is where $f'(x) = 0$ ← horizontal tangents

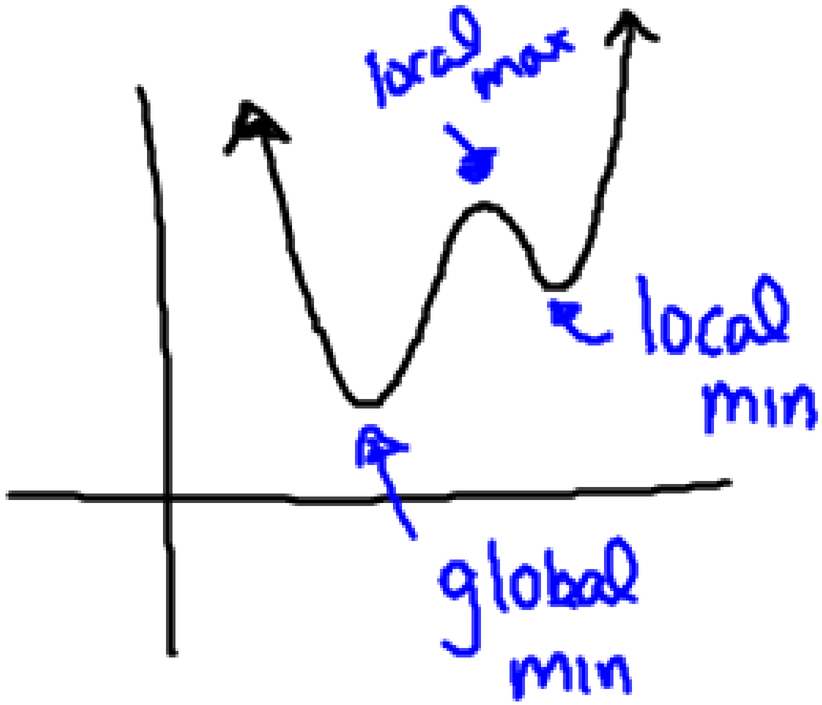
This can happen in two situations:

1) The graph has a turning point (local maximum or minimum)




global
minimum


no global
max/min


local max
local
min
global
min

2) The graph has an inflection

$f(x) = (x-3)^3 + 1$

$f'(x) = 3(x-3)^2(1) + 0$

$f'(x) = 3(x-3)^2$

$0 = 3(x-3)^2$

$0 = (x-3)^2$

$x = 3$

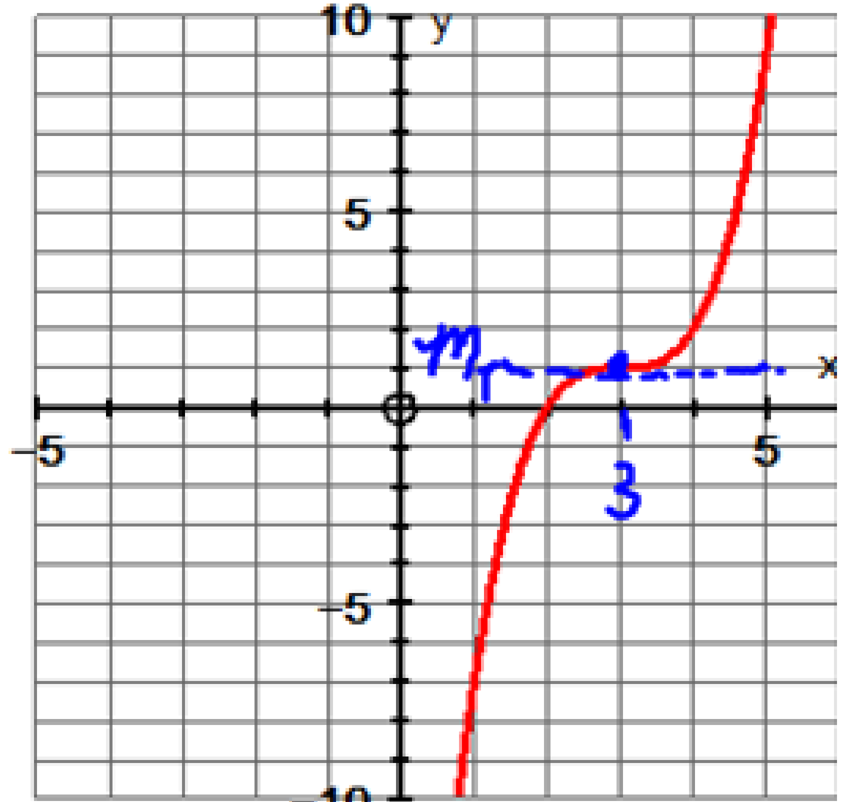
Sign diagram for $f'(x)$

+ | +

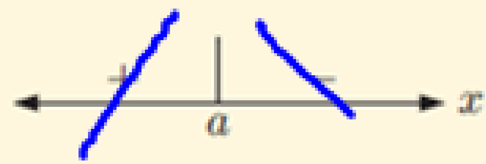
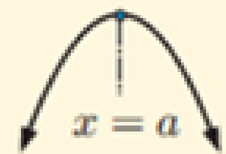
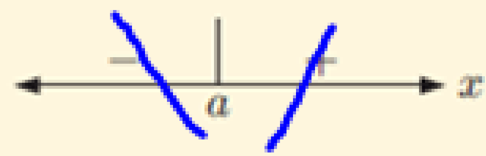
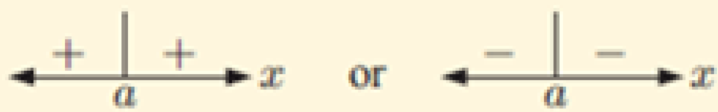
0 3

test $x=0$

$x=4$



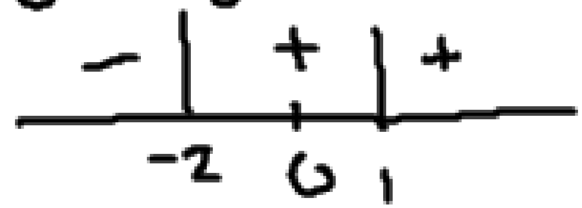
Sign Diagrams and Stationary Points

Stationary point where $f'(a) = 0$	*Sign diagram of* $f'(x)$ *near* $x = a$	*Shape of curve near* $x = a$
local maximum		
local minimum		
stationary inflection		

Example: Find and classify all stationary points. Sketch the function, labelling all important features (#2h pg 400)

$$f(x) = x^4 - 6x^2 + 8x - 3$$

sign diagram for $f'(x)$



test $x=0$

$x=2$

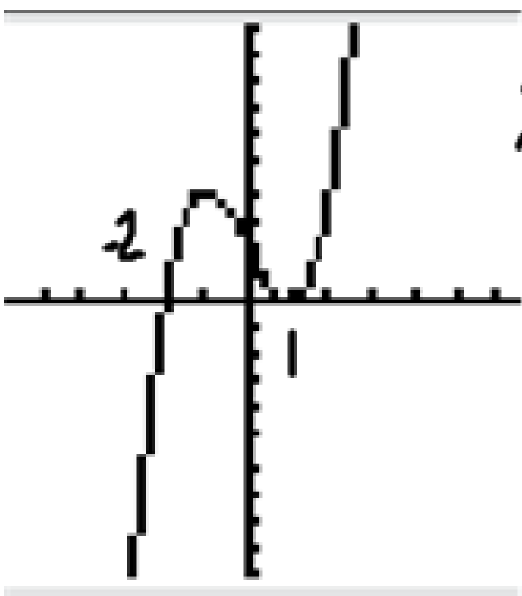
$x=-3$

$f'(x) = 4x^3 - 12x + 8$

Stationary pts $f'(x) = 0$

$0 = 4x^3 - 12x + 8$

$0 = x^3 - 3x + 2$



$x=1, x=-2$

$0 = (x-1)^2(x+2)$

at $x=-2$ we have a local min

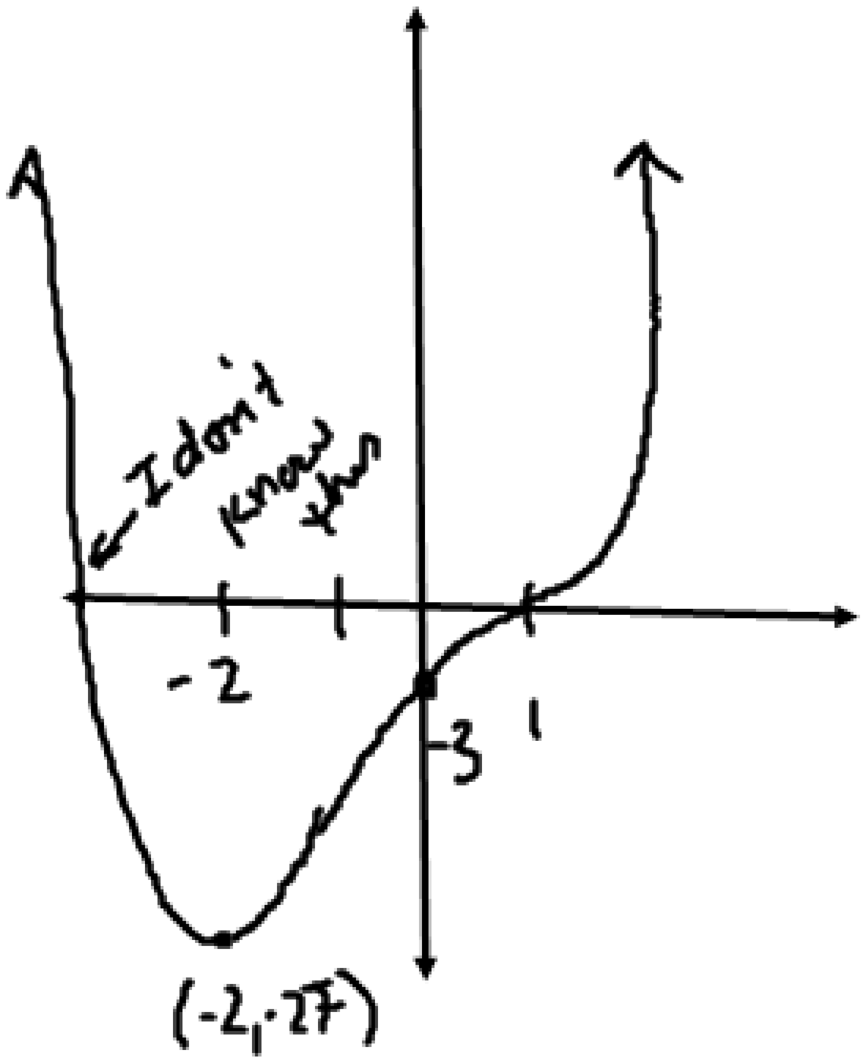
$f(-2) = (-2)^4 - 6(-2)^2 + 8(-2) - 3$

$f(-2) = -27$

at $x=1$ we have an inflection pt

$f(1) = (1)^4 - 6(1)^2 + 8(1) - 3$

$= 0$



← I don't know this
-2
1
-3
(-2, -27)

Example: $f(x) = x^3 - 6x^2 + ax + b$ has a stationary point at (2,1)

$\to m_T = 0$

A) Find the values of a and b

B) Find the position and nature of the stationary point

$f(2) = (2)^3 - 6(2)^2 + a(2) + b$

$1 = 8 - 24 + 2a + b$

$1 = -16 + 2a + b$

$17 = 2a + b$

$b = 17 - 2a$

$b = 17 - 2(12)$

$= 17 - 24$

$\boxed{b = -7}$

$f'(x) = 3x^2 - 12x + a$

$f'(2) = 3(2)^2 - 12(2) + a$

$0 = 12 - 24 + a$

$\boxed{12 = a}$

$f(x) = x^3 - 6x^2 + 12x - 7$

$f'(x) = 3x^2 - 12x + 12$

$x = 2$ is an inflectn pt

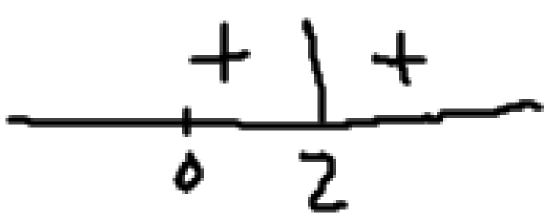
$0 = 3x^2 - 12x + 12$

$0 = x^2 - 4x + 4$

$0 = (x-2)(x-2)$

$x = 2$ repeated

sign diagram for $f'(x)$



EXERCISE 16C

1 The tangents at points A, B and C are horizontal.

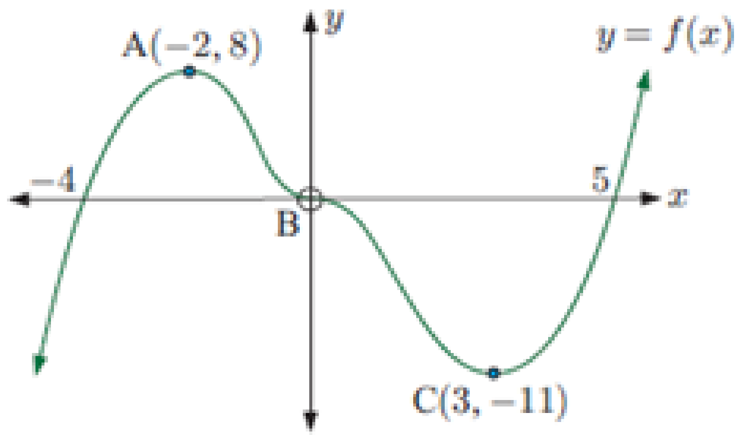
a Classify points A, B and C.

b Draw a sign diagram for the gradient function $f'(x)$ for all x.

c State intervals where $y = f(x)$ is:

i increasing **ii** decreasing.

d Draw a sign diagram for $f(x)$ for all x.



2 For each of the following functions, find and classify any stationary points. Sketch the function, showing all important features.

a $f(x) = x^2 - 2$

b $f(x) = x^3 + 1$

c $f(x) = x^3 - 3x + 2$

d $f(x) = x^4 - 2x^2$

e $f(x) = x^3 - 6x^2 + 12x + 1$

f $f(x) = \sqrt{x} + 2$

g $f(x) = x - \sqrt{x}$

h $f(x) = x^4 - 6x^2 + 8x - 3$

i $f(x) = 1 - x\sqrt{x}$

j $f(x) = x^4 - 2x^2 - 8$

3 At what value of x does the quadratic function $f(x) = ax^2 + bx + c$, $a \neq 0$, have a stationary point? Under what conditions is the stationary point a local maximum or a local minimum?

4 Find the position and nature of the stationary point(s) of:

a $y = xe^{-x}$

b $y = x^2e^x$

c $y = \dfrac{e^x}{x}$

d $y = e^{-x}(x + 2)$

5 $f(x) = 2x^3 + ax^2 - 24x + 1$ has a local maximum at $x = -4$. Find a.

6 $f(x) = x^3 + ax + b$ has a stationary point at $(-2, 3)$.

a Find the values of a and b.

b Find the position and nature of all stationary points.

7 Consider $f(x) = x \ln x$.

a For what values of x is $f(x)$ defined?

b Show that the global minimum value of $f(x)$ is $-\frac{1}{e}$.

8 For each of the following, determine the position and nature of the stationary points on the interval $0 \leqslant x \leqslant 2\pi$, then show them on a graph of the function.

a $f(x) = \sin x$ **b** $f(x) = \cos(2x)$ **c** $f(x) = \sin^2 x$

d $f(x) = e^{\sin x}$ **e** $f(x) = \sin(2x) + 2\cos x$

9 The cubic polynomial $P(x) = ax^3 + bx^2 + cx + d$ touches the line with equation $y = 9x + 2$ at the point $(0, 2)$, and has a stationary point at $(-1, -7)$. Find $P(x)$.

10 Find the greatest and least value of:

a $x^3 - 12x - 2$ for $-3 \leqslant x \leqslant 5$ **b** $4 - 3x^2 + x^3$ for $-2 \leqslant x \leqslant 3$

11 Show that $y = 4e^{-x}\sin x$ has a local maximum when $x = \frac{\pi}{4}$.

12 Prove that $\dfrac{\ln x}{x} \leqslant \dfrac{1}{e}$ for all $x > 0$.

Hint: Let $f(x) = \dfrac{\ln x}{x}$ and find its greatest value.

13 Consider the function $f(x) = x - \ln x$.

a Show that the graph of $y = f(x)$ has a local minimum and that this is the only turning point.

b Hence prove that $\ln x \leqslant x - 1$ for all $x > 0$.