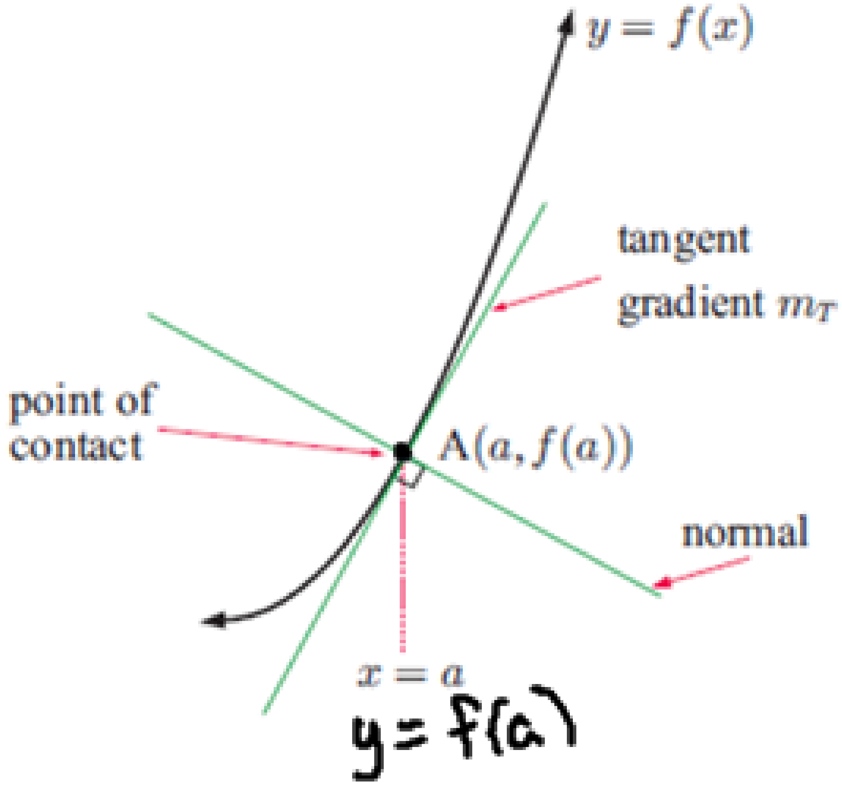
Chapter 16

Properties of curves

Syllabus reference: 6.1, 6.3

Contents:

16A – Tangents and Normals



If A is a point with x-coordinate a, then the gradient of the tangent to the curve at this point is $f'(a) = m_T$

The tangent is a straight line – it's equation will be of the form $y = m_T x + b$.

The slope formula can also be used to find the equation of a line: $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

So $\underbrace{\dfrac{y - f(a)}{x - a}}_{\text{slope}} = f'(a)$ Or $y - f(a) = f'(a)(x - a)$

$\leftarrow y_1$

$\leftarrow x_1$

fancy way of saying eq'n of line $y = mx + b$

A **normal** to a curve is a line which is **perpendicular** to the tangent at the point of contact.

The gradient of the normal to the curve at $x = a$ is

$$m_N = -\frac{1}{f'(a)}$$

$$m_N = \frac{-1}{m_T}$$

Example: Find the equation of the tangent line of the curve $y = x^2 - 6x + 11$ at $x = 5$. Find the equation of the normal to the curve.

Step 1 : find y when $x = 5$

$y = (5)^2 - 6(5) + 11$
$= 25 - 30 + 11$
$y = 6$

Step 2 - find the derivative of y

$y' = 2x - 6$

Step 3 - find the slope of the tangent

$m_T = 2(5) - 6$
$= 4$

Step 4 - find eq'n of tangent line

$y = mx + b$
$6 = (4)(5) + b$
$6 = 20 + b$
$-14 = b$

$y = 4x - 14$

Normal line

$m_N = \frac{-1}{4}$

$y = mx + b$

Normal line $\rightarrow$ $y = -\frac{1}{4}x + \frac{29}{4}$

$m_N = \frac{-1}{4}$

$y = mx + b$

$6 = \left(\frac{-1}{4}\right)(5) + b$

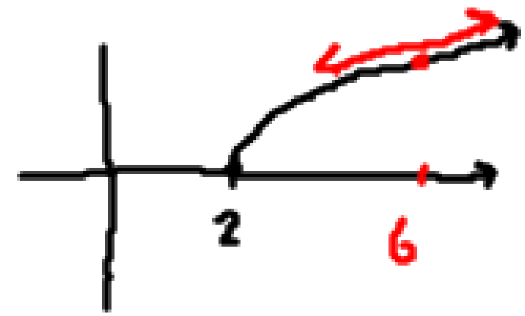
$6 = -\frac{5}{4} + b$

$6 + \frac{5}{4} = b$

$\frac{24}{4} + \frac{5}{4} = b$

$\frac{29}{4} = b$

Example: Find the equation of the tangent line of the function $y = \sqrt{x-2}$ at $x = 6$.



$y = \sqrt{6-2}$

$y = \sqrt{4}$

$y = 2$

$y = (x-2)^{\frac{1}{2}}$

$y' = \frac{1}{2}(x-2)^{-\frac{1}{2}}(1)$

$$y' = \frac{1}{2\sqrt{x-2}}$$

$$m_T = \frac{1}{2\sqrt{6-2}}$$

$$= \frac{1}{2\sqrt{4}}$$

$$= \frac{1}{4}$$

$y = mx + b$

$2 = \left(\frac{1}{4}\right)(6) + b$

$2 = \frac{3}{2} + b$

$\frac{4}{2} - \frac{3}{2} = b$

$\frac{1}{2} = b$

$$y = \frac{1}{4}x + \frac{1}{2}$$

Example: Find the equations of any horizontal tangents to $y = x^3 - 3x^2 + 5$.

$m_T = 0$

$y' = 3x^2 - 6x$

$0 = 3x^2 - 6x$

$0 = 3x(x-2)$

$3x = 0 \qquad x - 2 = 0$

$\boxed{x = 0} \qquad \boxed{x = 2}$

find the y value at each of those x positions

$\underline{x = 0}$

$y = (0)^3 - 3(0)^2 + 5$

$y = 5$

$\underline{x = 2}$

$y = (2)^3 - 3(2)^2 + 5$

$= 8 - 3(4) + 5$

$= 8 - 12 + 5$

$y = 1$

The eq'ns of the horizontal tangents are $y = 5$ and $y = 1$

EXERCISE 16A

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1 Find the equation of the tangent to:

a $y = x - 2x^2 + 3$ at $x = 2$

b $y = \sqrt{x} + 1$ at $x = 4$

c $y = x^3 - 5x$ at $x = 1$

d $y = \frac{4}{\sqrt{x}}$ at $(1, 4)$

e $y = \frac{3}{x} - \frac{1}{x^2}$ at $(-1, -4)$

f $y = 3x^2 - \frac{1}{x}$ at $x = -1$.

2 Find the equation of the normal to:

a $y = x^2$ at the point $(3, 9)$

b $y = x^3 - 5x + 2$ at $x = -2$

c $y = \frac{5}{\sqrt{x}} - \sqrt{x}$ at the point $(1, 4)$

d $y = 8\sqrt{x} - \frac{1}{x^2}$ at $x = 1$.

3 a Find the equations of the horizontal tangents to $y = 2x^3 + 3x^2 - 12x + 1$.

b Find the points of contact where horizontal tangents meet the curve $y = 2\sqrt{x} + \frac{1}{\sqrt{x}}$.

c Find k if the tangent to $y = 2x^3 + kx^2 - 3$ at the point where $x = 2$ has gradient 4.

d Find the equation of another tangent to $y = 1 - 3x + 12x^2 - 8x^3$ which is parallel to the tangent at $(1, 2)$.

4 a Consider the curve $y = x^2 + ax + b$ where a and b are constants. The tangent to this curve at the point where $x = 1$ is $2x + y = 6$. Find the values of a and b.

b Consider the curve $y = a\sqrt{x} + \frac{b}{\sqrt{x}}$ where a and b are constants. The normal to this curve at the point where $x = 4$ is $4x + y = 22$. Find the values of a and b.

c Show that the equation of the tangent to $y = 2x^2 - 1$ at the point where $x = a$, is $4ax - y = 2a^2 + 1$.