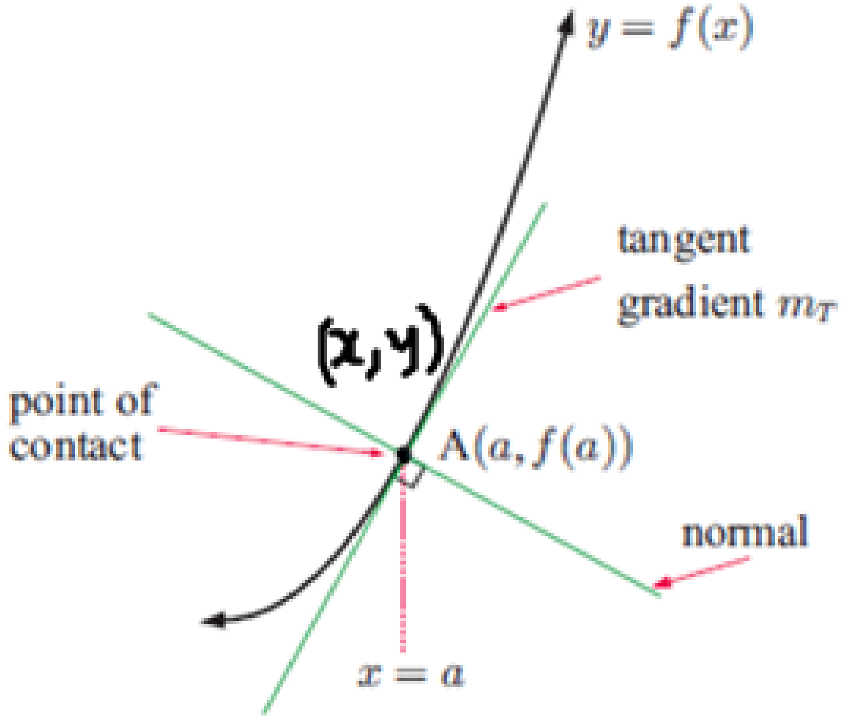
Chapter 16

Properties of curves

Syllabus reference: 6.1, 6.3

Contents:

16A – Tangents and Normals



If A is a point with x-coordinate a, then the gradient of the tangent to the curve at this point is $f'(a) = m_T$

The tangent is a straight line – it's equation will be of the form $y = m_T x + b$.

The slope formula can also be used to find the equation of a line: $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

slope slope

So $\dfrac{y - f(a)}{(x - a)} = f'(a)$ Or $y - \underline{f(a)} = \underline{f'(a)}(x - a)$ $y = m_T x + b$

A **normal** to a curve is a line which is **perpendicular** to the tangent at the point of contact.

The gradient of the normal to the curve at $x = a$ is

$$m_N = -\frac{1}{f'(a)}$$

$m_N = \dfrac{-1}{m_T}$

Example: Find the equation of the tangent line of the curve $y = x^2 - 6x + 11$ at $x = 5$. Find the equation of the normal to the curve.

Step1 : find derivative

$y' = 2x - 6$

Step2 : find slope of tangent at $x = 5$

$m_T = 2(5) - 6$

$m_T = 4$

Step3 find what y is at $x = 5$

$y = (5)^2 - 6(5) + 11$

$= 25 - 30 + 11$

$= 6$

Step4 - find eq'n of tangent line

$y = mx + b$

$6 = (4)(5) + b$

$6 = 20 + b$

$-14 = b$

$y = 4x - 14$

Step5 - find eq'n of Normal line

$m_N = \frac{-1}{4}$

$6 = (-\frac{1}{4})(5) + b$

$\frac{24}{4} + \frac{5}{4} = b$

$y = -\frac{1}{4}x + \frac{29}{4}$

$y - 6 = -\frac{1}{4}(x - 5)$

Example: Find the equations of any horizontal tangents to $y = x^3 - 3x^2 + 5$.

$m_T = 0$

$y' = 3x^2 - 6x$

$0 = 3x^2 - 6x$

$0 = 3x(x-2)$

$3x = 0 \qquad x - 2 = 0$

$x = 0 \qquad x = 2$

at $x=0$

$y = 0^3 - 3(0)^2 + 5$

$y = 5$

$y = mx + b$

$y = b$

$\boxed{y = 5}$

at $x = 2$

$y = 2^3 - 3(2)^2 + 5$

$= 8 - 12 + 5$

$= 1$

$\boxed{y = 1}$

✓ horizontal tangent lines

Example: Find the coordinates of the point(s) where the tangent to $y = x^3 - x^2 + 2x - 3$ at (1,-1) meets the curve again.

① Find eq'n of tangent line

$y' = 3x^2 - 2x + 2$

$m_T = 3(1)^2 - 2(1) + 2$

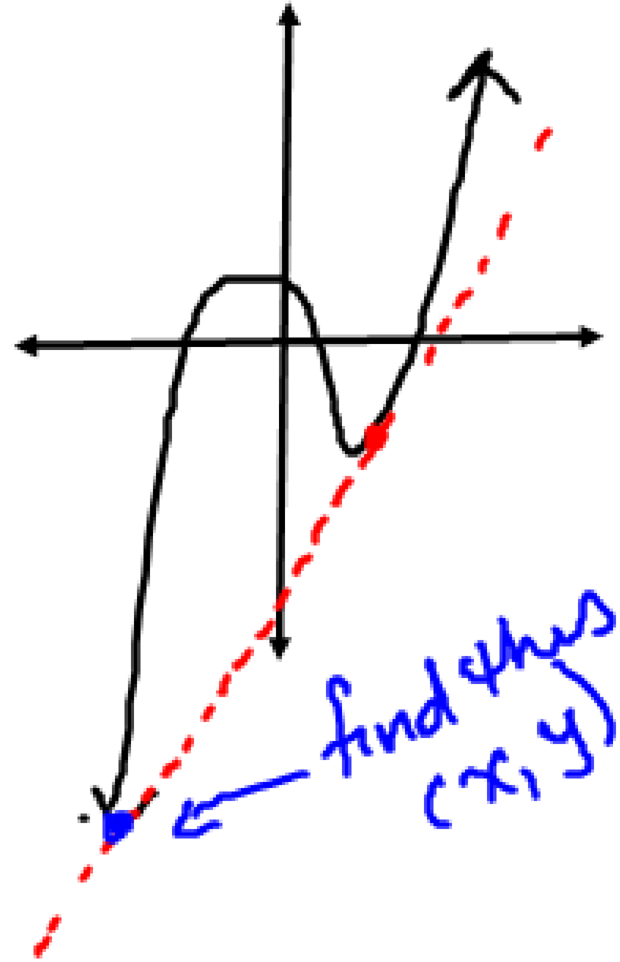
$= 3$

$y = m_T x + b$

$-1 = (3)(1) + b$

$-4 = b$

$\boxed{y = 3x - 4}$



② Find where the two curves intersect.

$$x^3 - x^2 + 2x - 3 = 3x - 4$$

$$x^3 - x^2 + 2x - 3x - 3 + 4 = 0$$

$$\underbrace{x^3 - x^2} - \underbrace{x + 1} = 0$$ factor by grouping

$$x^2(x-1) - 1(x-1) = 0$$

$$(x-1)(x^2-1) = 0$$

$$(x-1)(x+1)(x-1) = 0$$

$x = 1$ $\quad$ $x = -1$

already given this

can substitute this into either curve...

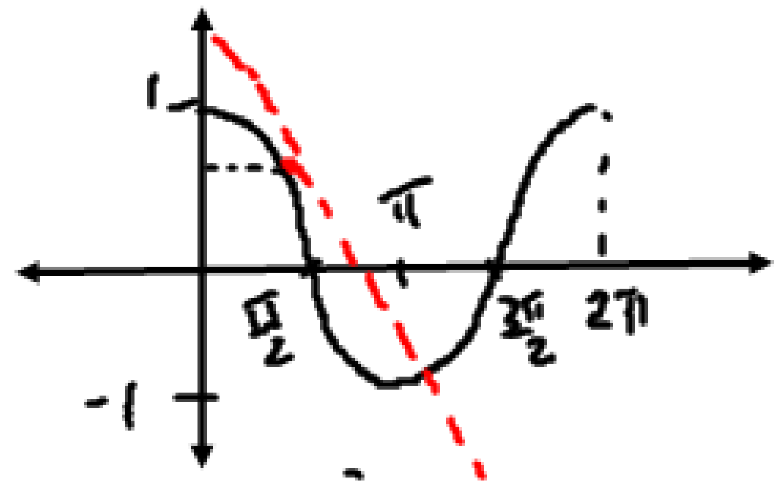
$$y = 3x - 4$$

$$y = 3(-1) - 4$$

$$y = -7$$

where tangent meets curve again → $(-1, -7)$ ☺

Example: Find the equation of the tangent to $y = \cos x$ at the point where $x = \frac{\pi}{3}$.



① find y coord. pt

$y = \cos x$

$y = \cos\left(\frac{\pi}{3}\right)$

$= \frac{1}{2}$

② slope of tangent line

$y' = -\sin x$

$m_T = -\sin\left(\frac{\pi}{3}\right)$

$m_T = -\frac{\sqrt{3}}{2}$

③ eqn of tangent line

$y = mx + b$

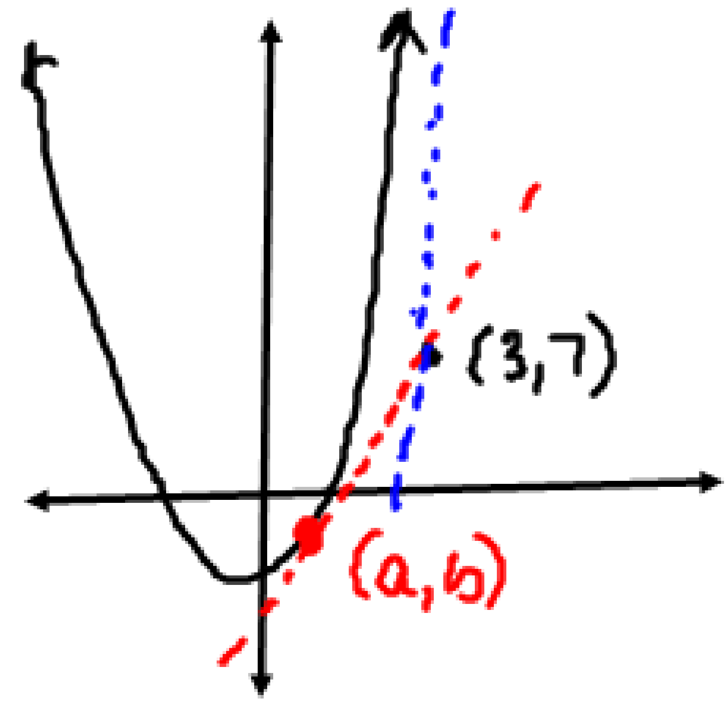
$\frac{1}{2} = \left(-\frac{\sqrt{3}}{2}\right)\left(\frac{\pi}{3}\right) + b$

$\frac{3}{6} + \frac{\pi\sqrt{3}}{6} = b$

$b = \frac{3 + \pi\sqrt{3}}{6}$

$y = -\frac{\sqrt{3}}{2}x + \left(\frac{3 + \pi\sqrt{3}}{6}\right)$

Example: Find the equations of the tangents to $\overset{f(x)}{\cancel{y}} = x^2 + 3x - 2$ from the external point (3, 7).



① find slope of tangent line at $x = a$

$f'(x) = 2x + 3$

$f'(a) = 2a + 3$

let (a, b) represent a point on $f(x)$ so that $f(a) = a^2 + 3a - 2$

$b = a^2 + 3a - 2$

② Find the eq'n of the tangent at (a, b)

$m = \dfrac{y_2 - y_1}{x_2 - x_1}$ $\qquad 2a + 3 = \dfrac{y - b}{x - a}$

$$2a+3=\frac{y-b}{x-a}$$

$$y-b=(x-a)(2a+3)$$

$$y-b=2ax+3x-2a^2-3a$$

$$y-(a^2+3a-2)=2ax+3x-2a^2-3a$$

plug $(3,7)$

$$7-a^2-3a+2=2a(3)+3(3)-2a^2-3a$$

$$9-a^2-3a=6a+9-2a^2-3a$$

$$\cancel{9}-a^2-\cancel{3a}-6a-\cancel{9}+2a^2+\cancel{3a}=0$$

$$a^2-6a=0$$

$$a^2-6a=0$$

$$a(a-6)=0$$

$$a=0 \quad a=6$$

When $a=0$

$b=a^2+3a-2$

$b=-2$

$(0,-2)$
(a,b)

eq'n of tangent

$\frac{y-b}{x-a}=2a+3$

$\frac{y-(-2)}{x-0}=2(0)+3$

$y+2=3x$

$\boxed{y=3x-2}$

When $a=6$

$b=(6)^2+3(6)-2$

$b=52$

$\frac{y-52}{x-6}=2(6)+3$

⋮

$\boxed{y=15x-38}$

EXERCISE 16A

1 Find the equation of the tangent to:

a $y = x - 2x^2 + 3$ at $x = 2$

b $y = \sqrt{x} + 1$ at $x = 4$

c $y = x^3 - 5x$ at $x = 1$

d $y = \frac{4}{\sqrt{x}}$ at $(1, 4)$

e $y = \frac{3}{x} - \frac{1}{x^2}$ at $(-1, -4)$

f $y = 3x^2 - \frac{1}{x}$ at $x = -1$.

2 Find the equation of the normal to:

a $y = x^2$ at the point $(3, 9)$

b $y = x^3 - 5x + 2$ at $x = -2$

c $y = \frac{5}{\sqrt{x}} - \sqrt{x}$ at the point $(1, 4)$

d $y = 8\sqrt{x} - \frac{1}{x^2}$ at $x = 1$.

3 a Find the equations of the horizontal tangents to $y = 2x^3 + 3x^2 - 12x + 1$.

b Find the points of contact where horizontal tangents meet the curve $y = 2\sqrt{x} + \frac{1}{\sqrt{x}}$.

c Find k if the tangent to $y = 2x^3 + kx^2 - 3$ at the point where $x = 2$ has gradient 4.

d Find the equation of another tangent to $y = 1 - 3x + 12x^2 - 8x^3$ which is parallel to the tangent at $(1, 2)$.

4 a Consider the curve $y = x^2 + ax + b$ where a and b are constants. The tangent to this curve at the point where $x = 1$ is $2x + y = 6$. Find the values of a and b.

b Consider the curve $y = a\sqrt{x} + \frac{b}{\sqrt{x}}$ where a and b are constants. The normal to this curve at the point where $x = 4$ is $4x + y = 22$. Find the values of a and b.

c Show that the equation of the tangent to $y = 2x^2 - 1$ at the point where $x = a$, is $4ax - y = 2a^2 + 1$.

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8 Find the equation of:

a the tangent to the function $f : x \mapsto e^{-x}$ at the point where $x = 1$

b the tangent to $y = \ln(2 - x)$ at the point where $x = -1$

c the normal to $y = \ln\sqrt{x}$ at the point where $y = -1$.

9 Show that the curve with equation $y = \dfrac{\cos x}{1 + \sin x}$ does not have any horizontal tangents.

10 Find the equation of:

a the tangent to $y = \sin x$ at the origin

b the tangent to $y = \tan x$ at the origin

c the normal to $y = \cos x$ at the point where $x = \frac{\pi}{6}$

d the normal to $y = \dfrac{1}{\sin(2x)}$ at the point where $x = \frac{\pi}{4}$.

11 a Find where the tangent to the curve $y = x^3$ at the point where $x = 2$, meets the curve again.

b Find where the tangent to the curve $y = -x^3 + 2x^2 + 1$ at the point where $x = -1$, meets the curve again.

12 Consider the function $f(x) = x^2 + \frac{4}{x^2}$.

a Find $f'(x)$.

b Find the values of x at which the tangent to the curve is horizontal.

c Show that the tangents at these points are the same line.

13 The tangent to $y = x^2 e^x$ at $x = 1$ cuts the x and y-axes at A and B respectively. Find the coordinates of A and B.

14 **a** Find the equation of the tangent to $y = x^2 - x + 9$ at the point where $x = a$. Hence, find the equations of the two tangents from $(0, 0)$ to the curve. State the coordinates of the points of contact.

b Find the equations of the tangents to $y = x^3$ from the external point $(-2, 0)$.

c Find the equation of the normal to $y = \sqrt{x}$ from the external point $(4, 0)$.
Hint: There is no normal at the point where $x = 0$, as this is the endpoint of the function.

15 Find the equation of the tangent to $y = e^x$ at the point where $x = a$.
Hence, find the equation of the tangent to $y = e^x$ which passes through the origin.

16 Consider $f(x) = \dfrac{8}{x^2}$.

a Sketch the graph of the function.

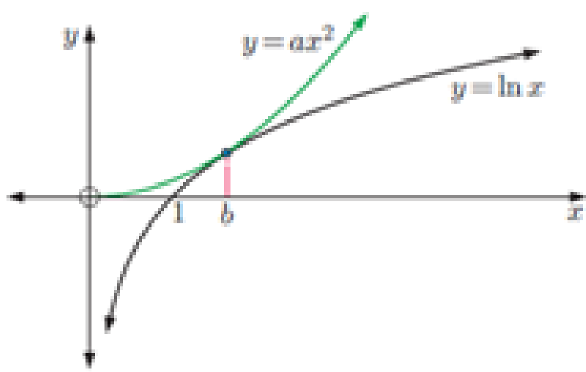
b Find the equation of the tangent at the point where $x = a$.

c If the tangent in **b** cuts the x-axis at A and the y-axis at B, find the coordinates of A and B.

d Find the area of triangle OAB and discuss the area of the triangle as $a \to \infty$.

17 Find, correct to 2 decimal places, the angle between the tangents to $y = 3e^{-x}$ and $y = 2 + e^x$ at their point of intersection.

18 A quadratic of the form $y = ax^2$, $a > 0$, touches the logarithmic function $y = \ln x$ as shown.



a If the x-coordinate of the point of contact is b, explain why $ab^2 = \ln b$ and $2ab = \dfrac{1}{b}$.

b Deduce that the point of contact is $(\sqrt{e}, \frac{1}{2})$.

c Find the value of a.

d Find the equation of the common tangent.