ch16A

Example: Find the coordinates of the point(s) where the tangent to $y = x^3 - x^2 + 2x - 3$ at (1,-1) meets the curve again.

① Find Eq'n of tangent line

$y' = 3x^2 - 2x + 2$

$m_T = 3(1)^2 - 2(1) + 2$
$= 3$

$y = mx + b$

$-1 = (3)(1) + b$

$-1 = 3 + b$

$-4 = b$

$\boxed{y = 3x - 4}$

② Find points of intersections

$x^3 - x^2 + 2x - 3 = 3x - 4$

$\underbrace{x^3 - x^2}\underbrace{- x + 1} = 0$ factor by grouping

$x^2(x-1) - 1(x-1) = 0$

$(x-1)(x^2-1) = 0$

$(x-1)(x+1)(x-1) = 0$

$x = 1$ and $x = -1$

they already meet at $(1,-1)$

they will meet again at $x=-1$

$\rightarrow$ can use either eq'n to find y value

$y = 3x - 4$

$y = 3(-1) - 4$

$y = -7$

$(-1,-7)$

Example: Find the equation of the normal to $y = \ln(2 - x)$ at the point where $x = -1$.

→ Find Slope of tangent

$y' = \frac{1}{2-x}(-1)$

$y' = \frac{-1}{2-x}$

$m_T = \frac{-1}{2-(-1)}$

$= \frac{-1}{3}$

$m_N = 3$

When $x = -1$

$y = \ln(2-(-1))$

$y = \ln 3$

$y = mx + b$

$\ln 3 = (3)(-1) + b$

$\ln 3 = -3 + b$

$\ln 3 + 3 = b$

$y = 3x + (\ln 3 + 3)$

Example: Find the equation of the tangent to $y = \cos x$ at the point where $x = \frac{\pi}{3}$.

$y = \cos\left(\frac{\pi}{3}\right)$

$y = \frac{1}{2}$

$y = \cos x$

$y' = -\sin x$

$m_T = -\sin\left(\frac{\pi}{3}\right)$

$m_T = -\left(\frac{\sqrt{3}}{2}\right)$

$m_T = \frac{-\sqrt{3}}{2}$

$y = mx + b$

$\frac{1}{2} = \frac{-\sqrt{3}}{2}\left(\frac{\pi}{3}\right) + b$

$\frac{3}{6} = \frac{-\pi\sqrt{3}}{6} + b$

$\frac{3+\pi\sqrt{3}}{6} = b$

$y = -\frac{\sqrt{3}}{2}x + \left(\frac{3+\pi\sqrt{3}}{6}\right)$

Example: Find the equations of the tangents to $y = x^2 + 3x - 2$ from the external point (3, 7).

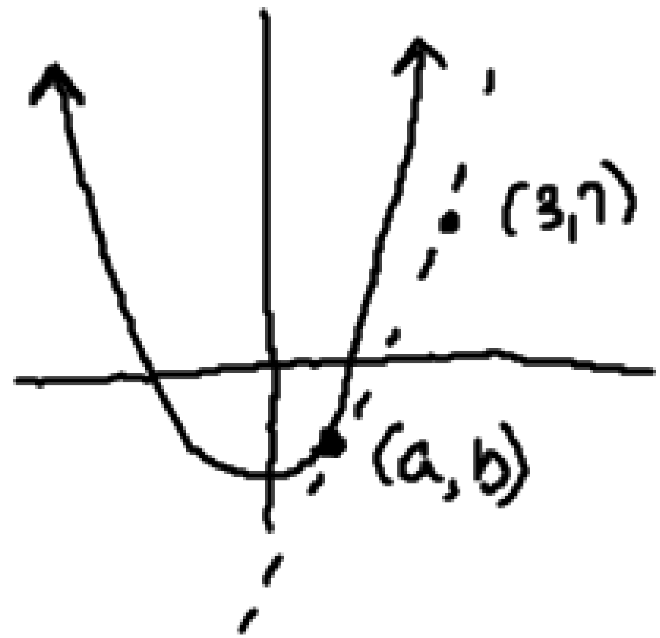
$f(x) = x^2 + 3x - 2$

$f'(x) = 2x + 3$

let (a, b) be a point on $f(x)$ so that

$f(a) = (a)^2 + 3a - 2$

$b = a^2 + 3a - 2$



Slope of the tangent at (a, b)

$f'(a) = 2(a) + 3$

$m_T = \dfrac{y_2 - y_1}{x_2 - x_1} \Rightarrow 2a + 3 = \dfrac{y - b}{x - a}$

$y - b = (x - a)(2a + 3)$ ← eq'n of the tangent

$y-b=(x-a)(2a+3)$ but $b=a^2+3a-2$

$y-(a^2+3a-2) = 2ax+3x-2a^2-3a$

↳ this mens passes through $(3, 7)$

$7-a^2-3a+2 = 2a(3)+3(3)-2a^2-3a$

$\cancel{9}-a^2-\cancel{3a}-6a\cancel{-9}+2a^2\cancel{+3a}=0$

$a^2-6a=0$

$a(a-6)=0$

$a=0,\ a=6$

$$b = a^2 + 3a - 2$$

When $a = 0$

$b = (0)^2 + 3(0) - 2 = -2$

$$\frac{y-b}{x-a} = 2a + 3$$

$$\frac{y+2}{x-0} = 2(0) + 3$$

$$\frac{y+2}{x} = 3$$

$$\boxed{y = 3x - 2}$$

When $a = 6$

$b = (6)^2 + 3(6) - 2$
$= 36 + 18 - 2$
$= 52$

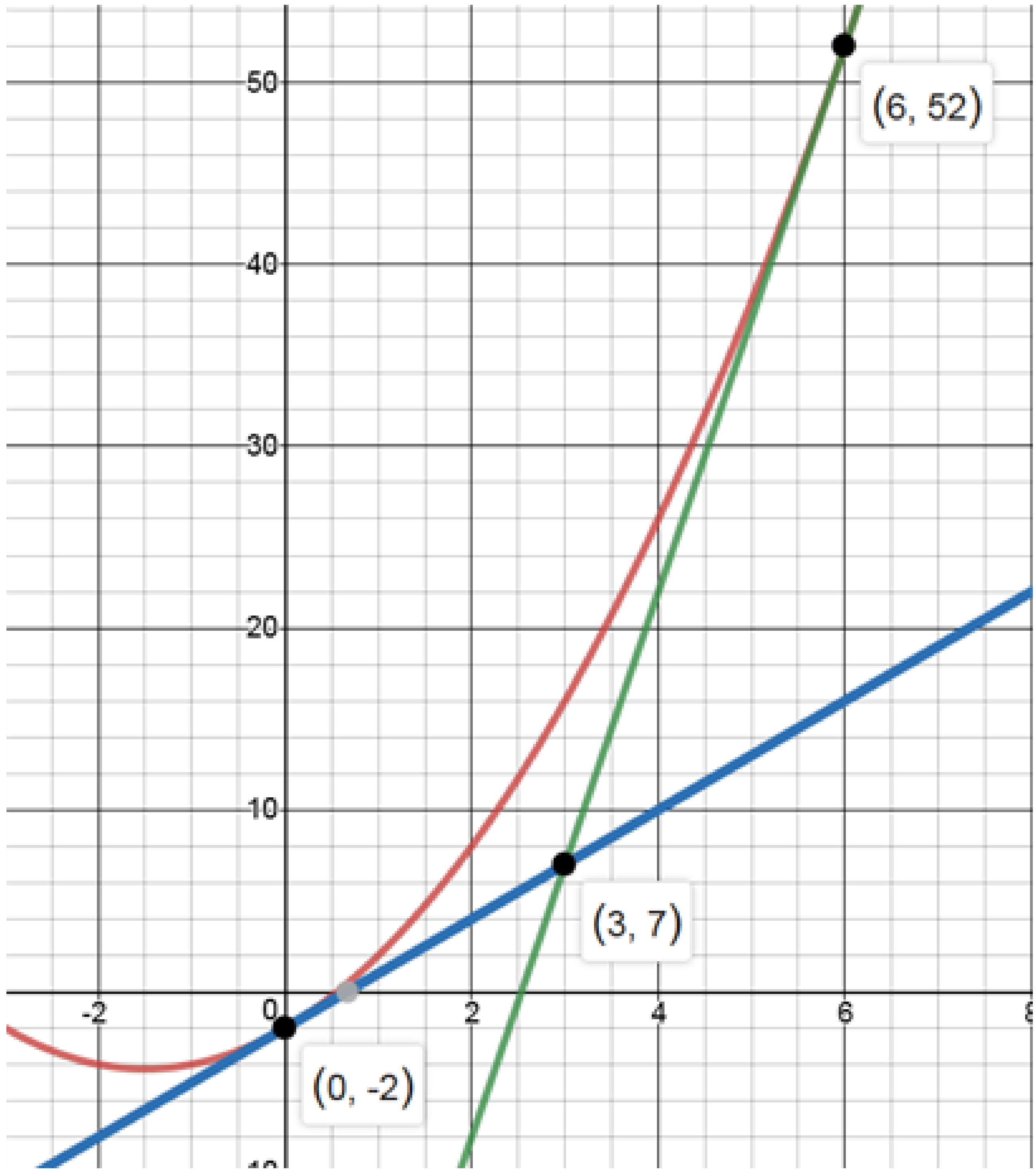
$$\frac{y-b}{x-a} = 2a + 3$$

$$\frac{y-52}{x-6} = 2(6) + 3$$

$$\frac{y-52}{x-6} = 15$$

$$y - 52 = 15(x-6)$$

$$y = 15x - 38$$



50
40
30
20
10
-2
0
2
4
6
8
(6, 52)
(3, 7)
(0, -2)

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8 Find the equation of:

- **a** the tangent to the function $f : x \mapsto e^{-x}$ at the point where $x = 1$
- **b** the tangent to $y = \ln(2 - x)$ at the point where $x = -1$
- **c** the normal to $y = \ln\sqrt{x}$ at the point where $y = -1$.

9 Show that the curve with equation $y = \dfrac{\cos x}{1 + \sin x}$ does not have any horizontal tangents.

10 Find the equation of:

- **a** the tangent to $y = \sin x$ at the origin
- **b** the tangent to $y = \tan x$ at the origin
- **c** the normal to $y = \cos x$ at the point where $x = \frac{\pi}{6}$
- **d** the normal to $y = \dfrac{1}{\sin(2x)}$ at the point where $x = \frac{\pi}{4}$.

11
- **a** Find where the tangent to the curve $y = x^3$ at the point where $x = 2$, meets the curve again.
- **b** Find where the tangent to the curve $y = -x^3 + 2x^2 + 1$ at the point where $x = -1$, meets the curve again.

12 Consider the function $f(x) = x^2 + \dfrac{4}{x^2}$.

- **a** Find $f'(x)$.
- **b** Find the values of x at which the tangent to the curve is horizontal.
- **c** Show that the tangents at these points are the same line.

13 The tangent to $y = x^2 e^x$ at $x = 1$ cuts the x and y-axes at A and B respectively. Find the coordinates of A and B.

14 **a** Find the equation of the tangent to $y = x^2 - x + 9$ at the point where $x = a$. Hence, find the equations of the two tangents from $(0, 0)$ to the curve. State the coordinates of the points of contact.

b Find the equations of the tangents to $y = x^3$ from the external point $(-2, 0)$.

c Find the equation of the normal to $y = \sqrt{x}$ from the external point $(4, 0)$.
Hint: There is no normal at the point where $x = 0$, as this is the endpoint of the function.

15 Find the equation of the tangent to $y = e^x$ at the point where $x = a$.
Hence, find the equation of the tangent to $y = e^x$ which passes through the origin.

16 Consider $f(x) = \dfrac{8}{x^2}$.

a Sketch the graph of the function.

b Find the equation of the tangent at the point where $x = a$.

c If the tangent in **b** cuts the x-axis at A and the y-axis at B, find the coordinates of A and B.

d Find the area of triangle OAB and discuss the area of the triangle as $a \to \infty$.

17 Find, correct to 2 decimal places, the angle between the tangents to $y = 3e^{-x}$ and $y = 2 + e^x$ at their point of intersection.

18 A quadratic of the form $y = ax^2$, $a > 0$, touches the logarithmic function $y = \ln x$ as shown.

a If the x-coordinate of the point of contact is b, explain why $ab^2 = \ln b$ and $2ab = \dfrac{1}{b}$.

b Deduce that the point of contact is $(\sqrt{e}, \frac{1}{2})$.

c Find the value of a.

d Find the equation of the common tangent.